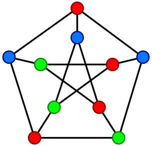
Graph coloring

Arnaud Casteigts

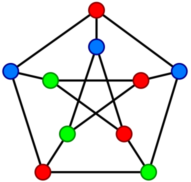


Part of Graph Algorithms (14X061)
Masters of Computer Science

University of Geneva

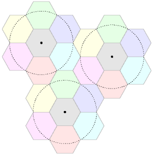
Graph coloring

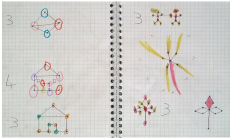
Goal: Assign a color to every vertex such that adjacent vertices have different colors.



Many applications (e.g. telecom)

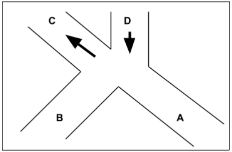
- Wireless communications
- In general, mutual exclusion, scheduling, ...
- Occupy a 5-y.o. kid

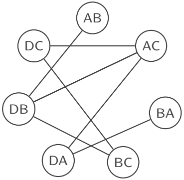


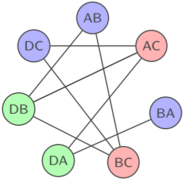


Graph coloring (2)

Example: traffic lights (credit: Eric Sopena)







1) Create a conflict graph G of the trajectories.

2) Color G

3) We obtain 3 color classes: $1 : \{AB, DC, BA\}$, $2 : \{DB, DA\}$, $3 : \{AC, BC\}$.

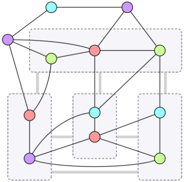
$\rightarrow$ All trajectories in the same class can have green light at the same time.

Further examples: time tables; altitude of aircrafts; anything to be optimized against conflicts.

Chromatic number $\chi(G)$

$\chi(G) =$ minimum number of colors needed in G.

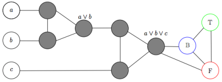
- At least the size of any clique in G
- Hadwiger's conjecture (1943):
  At most the size of a clique minor in G.



On the algorithmic side

Complexity of k-coloring

- 2-Col is linear (if and only if bipartite graph)
- 3-Col is NP-hard (reduction from SAT)
- k-Col is NP-hard (reduction from (k-1)-Col)

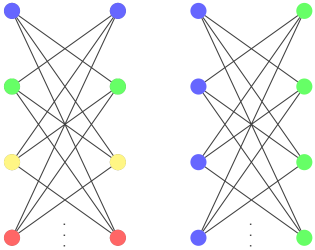


[Garey, Johnson, Stockmeyer, 1976]
(drawing: Yu Cheng)

The `First-Fit` algorithm

For each vertex in G:
Try color 1, then 2, then 3...

Very fast, but arbitrarily far from optimum
(if we pick the vertices in bad order)



$O(n)$ colors $O(1)$ colors

Still, essentially the best we can do!

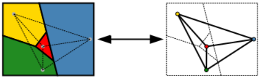
(No $n^{1-\epsilon}$ approx in polynomial time... [Zuckermann, 2007])

The four color theorem

Every planar graph is four colorable *(planar = can be drawn without crossing edges).*

Timeline:

- 1852 Francis Guthrie (botanist) notices that **four** colors are enough to color the map of England's counties.
- 1879 Kempe proves the conjecture.
- 1880 Tait formulates it in terms of **planar graphs**, and gives a different proof.
- 1890 Heawood finds a bug in Kempe's proof and adapts it to prove that **five** colors are enough.
- 1891 Petersen finds a bug in Tait's proof.
- 1960s Heesch starts using computers to search for a proof
- 1976 Appel and Haken **succeed!**
  $\rightarrow$ reduction from ∞ to 1834 possible configurations, all **checked by computer**.
- 1996 Robertson, Sanders, Seymour reduce it to 633 configurations.
- 2005 Gonthier certifies the proof using **Coq**. □



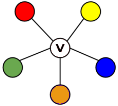
Proof of the 5 color theorem

Theorem: planar graphs are 5 colorable.

Warm-up: what about 6 colors first?

- Euler's formula: in planar graphs, $v - e + f = 2$ ($v = \#\text{vertices}$, $e = \#\text{edges}$, $f = \#\text{faces}$)
- Implies (not immediate) that every planar graph has a vertex of degree ≤ 5
- Recursive algorithm:
  1. Find a vertex v of degree ≤ 5
  2. Color $G \setminus v$
  3. Give an available color to v (guaranteed by its degree)

  Base case: If G has ≤ 5 vertices, give a different color to each vertex.



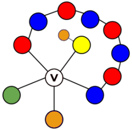
5 colors: Kempe's chains

Similar ideas as 6 colors, with an additional trick (by Kempe, 1879).

Step 2 becomes:
Color $G \setminus v$, then tweak the coloring so that the neighbors of v use at most 4 colors.

This is indeed always possible!

There must exist two colors which do not induce a connected component
$\rightarrow$ flip one of the components, one color is freed.



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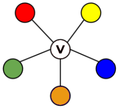

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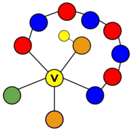
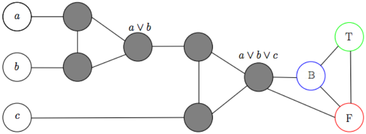
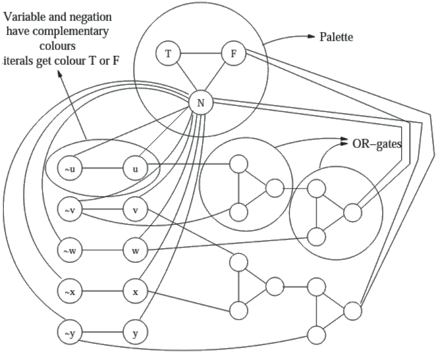


Reduction from 3-SAT to 3-COL



(credit: Lalla Mouatadid)

$$\varphi = (u \lor \neg v \lor w) \land (v \lor x \lor \neg y)$$

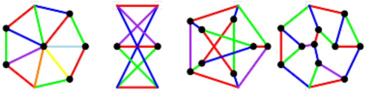


(credit: Igor Potapov)

Edge coloring

Sometimes, we prefer coloring the edges rather than the vertices.

Proper edge coloring: any two edges incident to a same vertex have different colors.



The chromatic index $\chi'(G)$ of a graph is the minimum number of colors needed.

Vizing's theorem: $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$ (where Δ is the largest degree)

Non-proper edge coloring

Remember the six friends (seens in exercises)

- Can we color the edges of K_5 (complete graph on 5 vertices) with two colors (know / don't know each other) such that no monochromatic triangle is created?

  Yes: outer cycle in one color, inner edges in the other color.

- Same question with K_6?

  No. Proof: Pick a vertex v. Without loss of generality, v has at least three incident edges with the same colors (say, color 1). Let v_1, v_2, v_3 be the corresponding neighbors.

  Now, consider the edges between vertices v_1, v_2, and v_3. Two possible cases:

  - At least one of these edges has color 1 $\implies$ this closes a triangle with v.
  - None of these edges has color 1 $\implies$ this makes a triangle with color 2.

Related problems and other comments

Clique and Independent Set

- Clique (G, k): Does G admit a clique of size k?
- Independent Set (G, k): Are there k vertices in G, none of them being neighbors?

$\rightarrow$ Clique can be used to obtain a lower bound on $\chi(G)$.
$\rightarrow$ Independent Set can be used in algorithms ($\rightarrow$ exercises).

Remark (already seen): Independent set $\leq_p$ Clique and Clique $\leq_p$ Independent set

Reduction (the same both ways): G admits a clique of size k iff $\overline{G}$ admits an independent set of size k.

Complexity of edge coloring?

Vizing's theorem: $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$ (where Δ is the largest degree)

$\rightarrow$ deciding between $\Delta(G)$ and $\Delta(G) + 1$ is NP-hard.