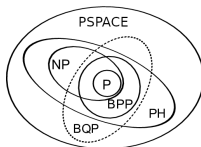


Basics of Computational Complexity

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Types of problems

- ▶ **Decision:** $\{0, 1\}^* \rightarrow \{0, 1\}$
Is the picture a cat?
Is there a path from A to B?
- ▶ **Search:** $\{0, 1\}^* \rightarrow \{0, 1\}^*$
Find the cat on the picture.
Give me a path from A to B.
- ▶ **Counting:** $\{0, 1\}^* \rightarrow \mathbb{N}$
How many cats are there on the picture?
How many paths are there from A to B?
- ▶ **Optimisation:** $\{0, 1\}^* \rightarrow \{0, 1\}^*$
Find the cutest cat in the picture.
Find a shortest path from A to B

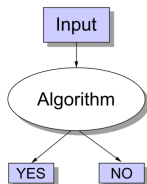


Decision problem (focus)

Functions F of type $\{0, 1\}^* \rightarrow \{0, 1\}$ (answer YES or NO)

Set of **positive instances** $\{x \in \{0, 1\}^* \mid F(x) = 1\}$ defines a **formal language** L

Solving a decision problem \equiv deciding the corresponding language



Gödel's letter to Von Neumann (1956)



Princeton, 20/II. 1956
Lieber Herr v. Neumann!
Ich habe mit größter Bedauern von Ihrer Erkrankung gehört. Die Nachricht kam mir ganz un erwartet. Morgenstern hatte mir zwar schon im Sommer von einem Schwächeanfall erzählt, den Sie einmal hatten, aber er meinte damals, dass das keine größere Bedeutung beizumessen sei. Wie ich höre, haben Sie sich in den letzten Monaten einer radikalen Behandlung unterzogen und hoffen, dass diese den gewünschten Erfolg hat, da er Ihnen jetzt besser geht. Ich hoffe u. wünsche Ihnen, dass Ihr Zustand sich bald noch weiter bessert u. dass die neuesten Erfindungen der Medizin, wenn möglich, zu einer vollständigen Heilung führen mögen.
Da Sie sich, wie ich höre, jetzt häufiger fühlen, möchte ich mir erlauben, Ihnen über ein mathematisches Problem zu schreiben, über das mich

Die Ansicht ist in der Theorie: Man kann offenbar leicht eine Turingmaschine konstruieren, welche von jeder Formel F des ersten Funktionalkalküls u. jeder natürl. Zahl n zu entscheiden gestattet, ob F einen Beweis der Länge n hat [Länge = Anzahl der Symbole]. Sei $\psi(F, n)$ die Anzahl der Schritte, die die Maschine dazu benötigt u. sei $\varphi(n) = \max_F \psi(F, n)$. Die Frage ist, wie rasch $\varphi(n)$ für eine optimale Maschine wächst. Man kann zeigen $\varphi(n) \geq Kn$. Wenn es nämlich eine Maschine mit $\varphi(n) \sim Kn$ (oder auch um $\sim Kn^2$) gäbe, hätte das Folgerungen von der größten Tragweite. Es würde nämlich offenbar bedeuten, dass man trotz der Unlösbarkeit des Entscheidungsproblems die Arbeit der Mathematiker bei ja-oder-nein-Fragen vollständig durch Maschinen ersetzen könnte. (folgenden

Gödel's letter to Von Neumann (1956)



I would like to allow myself to write you about a mathematical problem, of which your opinion would very much interest me. One can obviously construct a Turing machine, which for every formula F in first order predicate logic and every natural number n , allows one to decide if there is a proof of F of length n (length = number of symbols). Let $\Psi(F, n)$ be the number of steps the machine requires for this and let $\varphi(n) = \max_F \Psi(F, n)$.

The question is how fast $\varphi(n)$ grows for an optimal machine. (...) If there really were a machine with $\varphi(n) \sim n$ (or even $\sim n^2$), this would have consequences of the greatest importance. Namely, it would mean that (...) the mental work of a mathematician concerning Yes-or-No questions could be completely replaced by a machine.

(...) It would be interesting to know, for instance, the situation concerning the determination of primality of a number and how strongly in general the number of steps in finite combinatorial problems can be reduced with respect to simple exhaustive search.

... and here is computational complexity!

Computational complexity?

Amount of required resources for solving a problem.

What type of resources?

- ▶ Time (number of operations)
- ▶ Space (amount of memory)
- ▶ Non-determinism?
- ▶ Randomness?
- ▶ ...

Asymptotic point of view

- ▶ **Evolution** of these quantities as a function of the input size n , when $n \rightarrow \infty$
- ▶ Notations $O(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, $o(\cdot)$, $\omega(\cdot)$ (ignores **constant factors** and **dominated terms**)

Intuition \leq \geq $=$ $<$ $>$

Ex: $3n^2 + 5n + 4 = \Theta(n^2)$

- ▶ Some adjectives:

| | | | |
|--------------|--------------------|-------------|---|
| Constant | $\Theta(1)$ | Quadratic | $\Theta(n^2)$ |
| Logarithmic | $\Theta(\log n)$ | Exponential | $\Theta(2^n)$ or $\Theta(2^{n^{O(1)}})$ |
| Linear | $\Theta(n)$ | Factorial | $\Theta(n!)$ |
| Quasi-linear | $\Theta(n \log n)$ | Polynomial | $O(n^c) = n^{O(1)}$ |

In general, we are interested in the **worst case** (maximum over all possible instances of a problem).

Asymptotic notations (O , Ω , Θ)

| | True | False |
|--------------------------------------|------|-------|
| $4n^2 - 5n + 1 = O(n^2)$ | ✓ | |
| $4n^2 - 5n + 1 = \Theta(n^2)$ | ✓ | |
| $n \log n = \Omega(n)$ | ✓ | |
| $n \log n = O(n^2)$ | ✓ | |
| $n \log n = \Theta(\text{-----})?$ | N/A | N/A |
| $n \log n^2 = \Theta(\text{-----})?$ | N/A | N/A |
| $500 = \Theta(1)$ | ✓ | |
| $17n^2 + 3 = n^{O(1)}$ | ✓ | |
| $\sqrt{n} = O(n)$ | ✓ | |
| $n! = \Omega(2^n)$ | ✓ | |

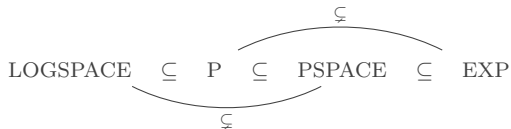
Time and space

Generic classes

- ▶ $\text{TIME}(f(n))$: Decision problems solvable in time $O(f(n))$ (regardless of space).
- ▶ $\text{SPACE}(f(n))$: Decision problems solvable in space $O(f(n))$ (regardless of time).

Well-known particular cases

| Name | Solvable in... | Definition |
|----------|-------------------|--------------------------|
| LOGSPACE | logarithmic space | $\text{SPACE}(\log n)$ |
| P | polynomial time | $\text{TIME}(n^{O(1)})$ |
| PSPACE | polynomial space | $\text{SPACE}(n^{O(1)})$ |
| EXP | exponential time | $\text{TIME}(2^n)$ |



The most important is P

Problems solvable “efficiently” (in time $n^{O(1)}$).

(robust / composable / realistic)

Class NP

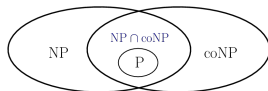
Several definitions, the simplest is:

NP: \exists **short proof** that the answer is YES (if it is YES) – *a.k.a.* positive certificate

coNP: \exists **short proof** that the answer is NO (if it is NO) – *a.k.a.* negative certificate

Short proof = **verifiable** in polynomial time

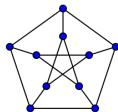
Observation: $P \subseteq NP$ and $P \subseteq coNP$ (the algorithm itself can be used as a verifier)



Some problems in NP (presumably not in P): 3-COLORING, CLIQUE, TSP, FACTORISATION, SAT, ...

Example: 3-COLORING

Can this graph be colored with 3 colors?



(certificate = the coloring itself)

I accept.

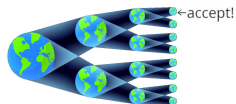


Verifier

Historical definition

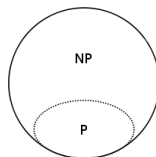
NP = Non-deterministic Polynomial time

Intuition: ability to “guess” the certificate (that it suffices to verify afterwards).



P versus NP

Does “easy to verify” imply “easy to solve”?
(Does $P = NP$?)



Relevance of the question

- ▶ Most practical problems are in NP.
If $P = NP$, all of them can be solved efficiently.
- ▶ Would it be a good news? Yes and no (cryptography).
- ▶ One of the 7 “problems of the millenium” (Clay fundation, \$1 M / pb), along with Riemann’s conjecture.

Philosophical implications?

- ▶ Math: can all humanly verifiable statement be settled by an algorithm?
- ▶ More generally: can we mechanize intuition?
- ▶ I can recognize a beautiful symphony, does it mean I could have composed it myself?
- ▶ *etc.* debate: formalization + how about $O(n^{100})$?

As of today, we don’t know the answer.

But most of the specialists believe $P \neq NP$.

NP-complete problems

Hardness and completeness

- ▶ NP-hard: problems **at least as hard** as any problem in NP
Can be shown through **reductions** among problems.
- ▶ NP-complete: both in NP and NP-hard

How to show that a problem is NP-hard?

→ find a problem that is already NP-hard and **reduce** it to your problem (in polynomial time).

Examples of NP-complete problems

- ▶ SAT: $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_4) \vee \dots$ (Cook, Levin'71)
- ▶ 3-COLORING, CLIQUE, SET COVER, HAMILTONIAN CYCLE, TSP,
- ▶ Thousands of problems...

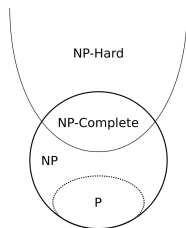
If **any** of these problems turns out to be in P, then they all are and $P = NP$.

If **any** of these problems turns out not to be in P, then none of them are and $P \neq NP$.

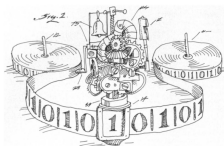
Important reminder

This theory focus on **worst case** complexity. Many instances from the real world are solvable in practice.

The real world is often nicer than an adversary.



What about AI?



V.S.



The computational complexity framework applies to AI, without distinction.

Many NP-complete problems are **easy on average**, so nothing precludes that AI can learn to solve them in most of the cases, but it won't do it in the **worst case**.

Some problems are **hard on average** and will remain out of reach for AI. Presumably, **FACTORING** is one example.

How about quantum computers?

BPP: *Bounded-error probabilistic polynomial time*

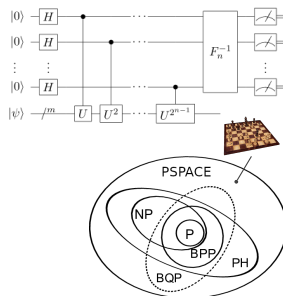
- ▶ Problems solvable in polynomial time by a **randomized algorithm** (can flip coins) with a probability of error lower than $1/2$.

BQP: *Bounded error quantum polynomial time*

- ▶ Problems solvable in polynomial time by a **quantum computer** with a probability of error lower than $1/2$.

What do we know?

- ▶ $P \subseteq BPP$ ($0 < 1/2$)
- ▶ $BPP \subseteq BQP$ (non-reversibility simulable in polynomial time)
- ▶ $BQP \subseteq PSPACE$ (Bernstein et Vazirani'97)
- ▶ $FACTURING \in BQP$ (Shor'94)
- ▶ How about BQP *versus* NP ? (expected incomparable)



(expected structure)

Is it expected that a quantum computer can solve NP-complete problems?

→ Unlikely (would contradict many plausible conjectures).

Some graph problems (decision version)

- ▶ SHORTEST PATH (G, u, v, k) : Does G admit a path of length at most k from u to v ? $\in P$
- ▶ LONGEST PATH (G, u, v, k) : Does G admit a path of length at least k from u to v ? NP-complete
- ▶ MATCHING (G, k) : Are there k edges in G that share no vertex in common $\in P$
- ▶ CLIQUE (G, k) : Does G admit a clique of size k ? NP-complete
- ▶ INDEPENDENT SET (G, k) : Are there k vertices in G , none of them being neighbors? NP-complete
- ▶ DOMINATING SET (G, k) : Is there a set of k vertices in G s.t. all nodes are either in the set or have a neighbor in the set? NP-complete
- ▶ VERTEX COVER (G, k) : Are there k vertices that collectively touch every edge? NP-complete
- ▶ COLORING (G, k) : Can G be properly colored with k colors? $\in P$ (if $k < 3$)
NP-complete (if $k \geq 3$)
- ▶ HAMILTONIAN CYCLE (G) : Does G admit a simple cycle that visits every vertex? NP-complete
- ▶ TSP (G, k) : Does G admit a simple cycle of cost $\leq k$ that visits every vertex? NP-complete
- ▶ GRAPH ISOMORPHISM (G_1, G_2) : Are G_1 and G_2 isomorphic? (i.e. structurally identical) NP-intermediate?

You'll play with some of these problems in exercises and we'll use them in subsequent classes.