Selected results in temporal graph theory

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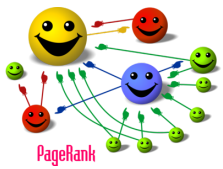
Séminaire “Groupes et Géométrie”,

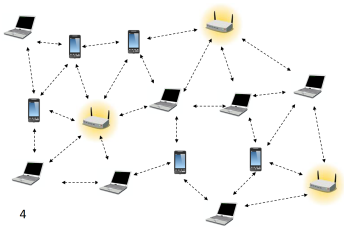
Section de mathématiques (UNIGE)

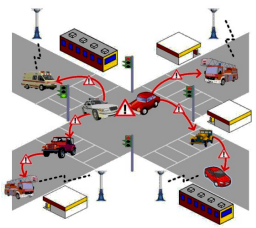
The world is dynamic...

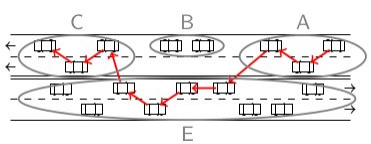
In technologies

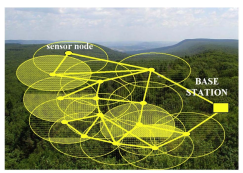


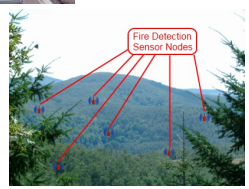


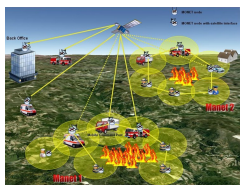


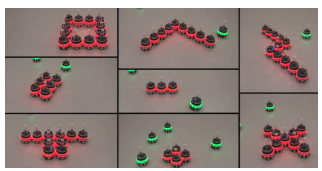








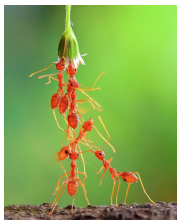


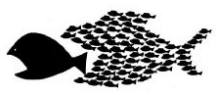


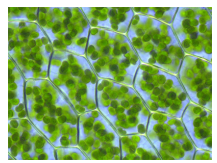
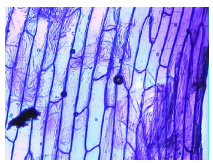

In nature





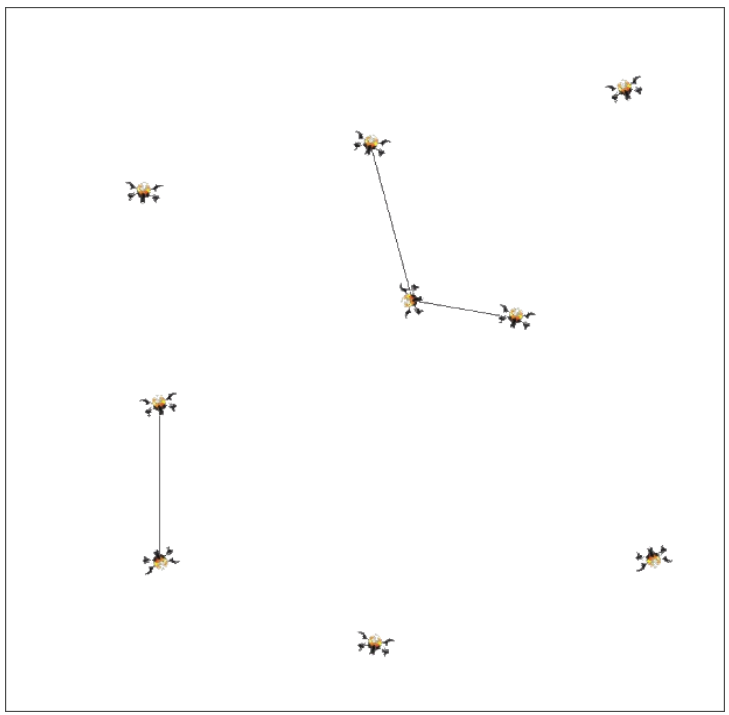




(Highly) dynamic networks?

Example of scenario



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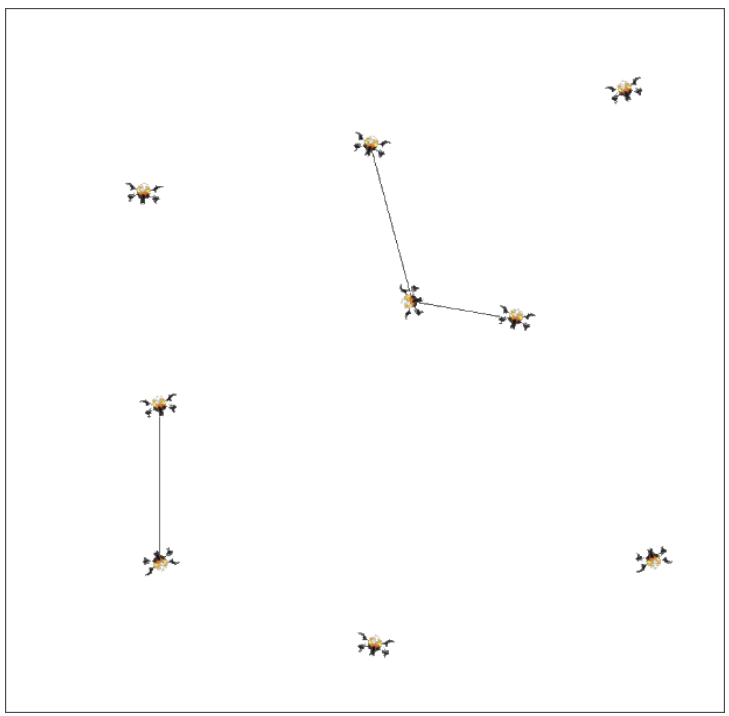
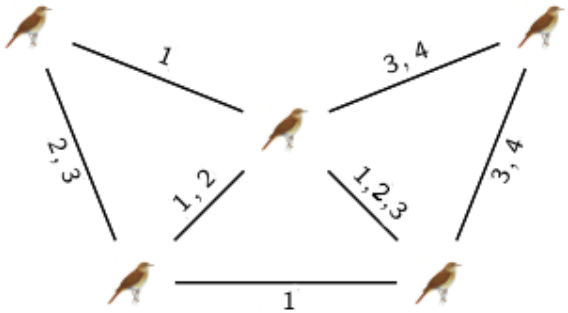
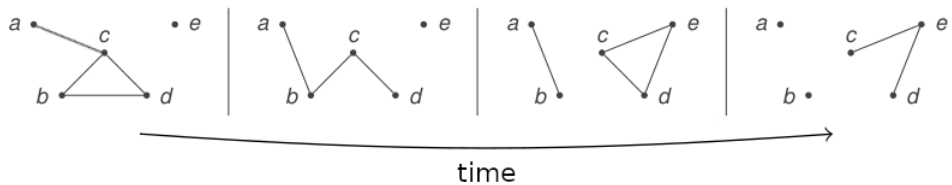


Modeling





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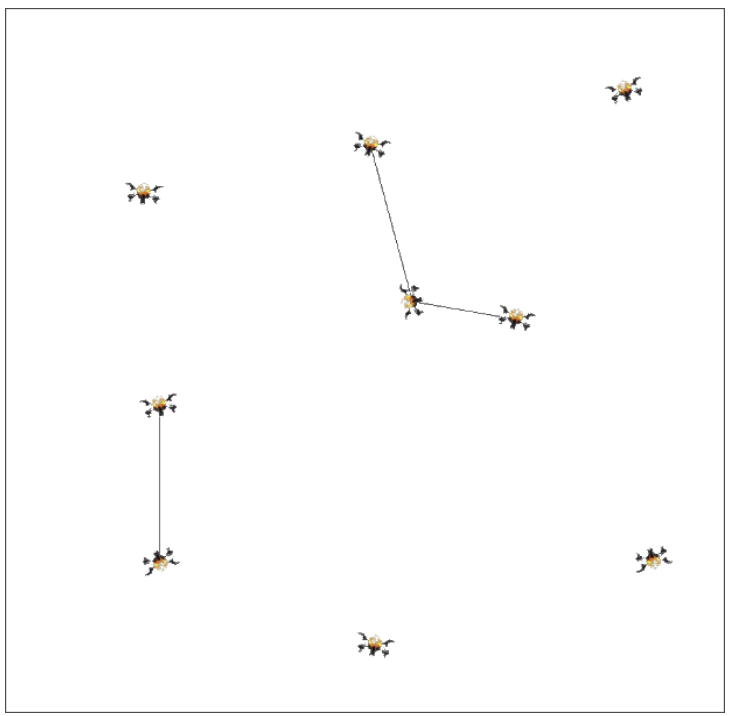

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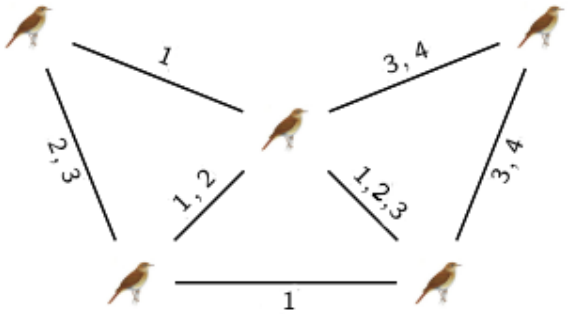

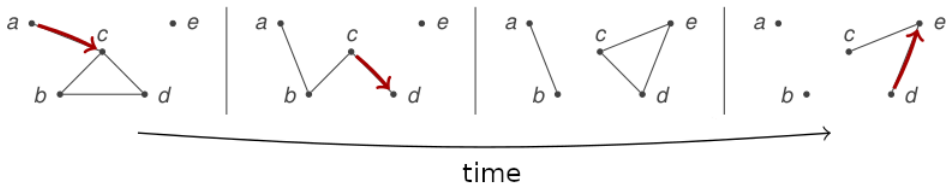

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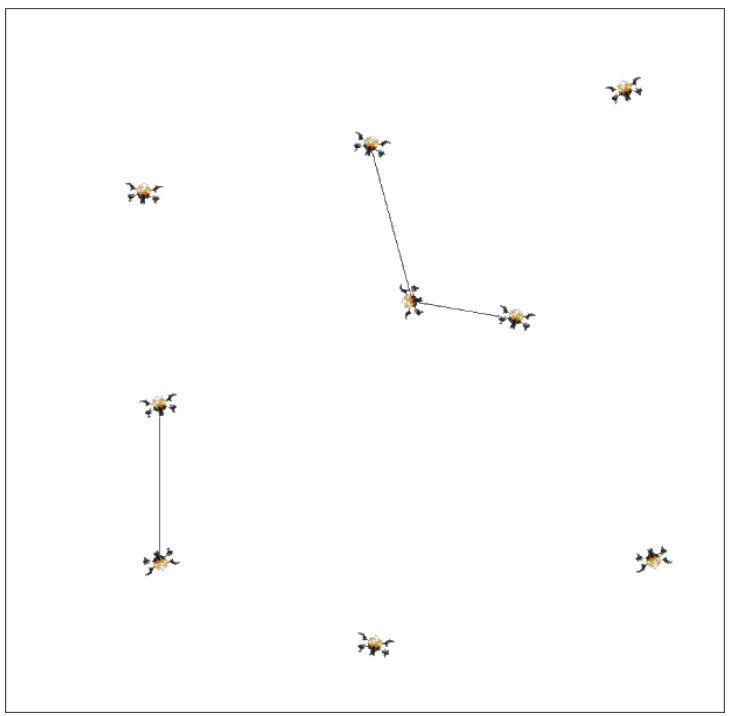

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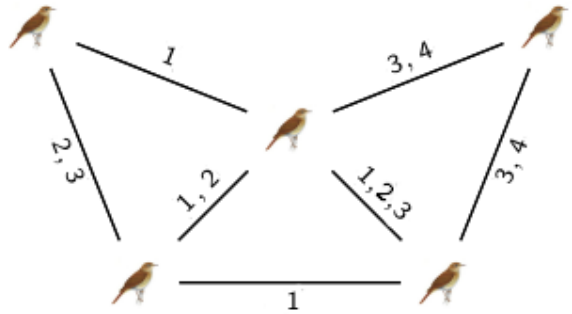

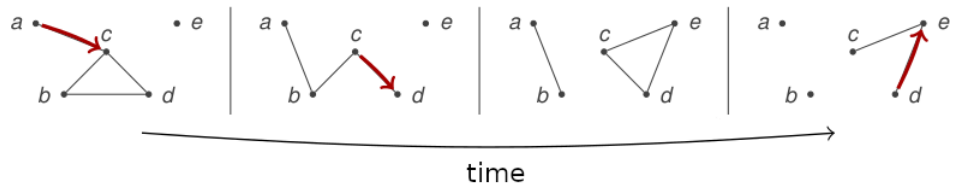

Properties:

- Temporal connectivity? $\mathcal{TC}$
- Repeatedly? $\mathcal{TC}^{\mathcal{R}}$
- Recurrent links? $\mathcal{E}^{\mathcal{R}}$
- In bounded time? $\mathcal{E}^{\mathcal{B}}$
- ...

→ Classes of temporal graphs

Temporal graphs

(*a.k.a.* time-varying, time-dependent, evolving, dynamic,...)

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns *time labels* to edges.

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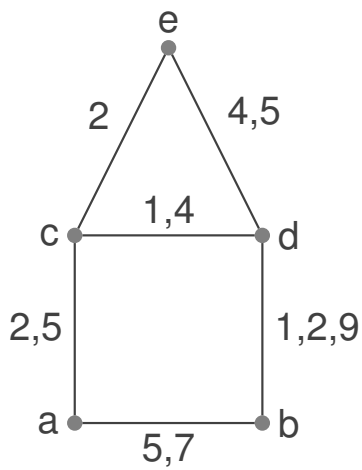

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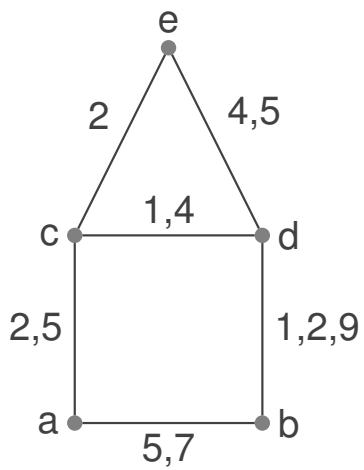

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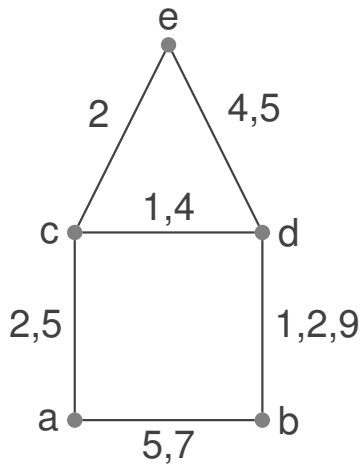

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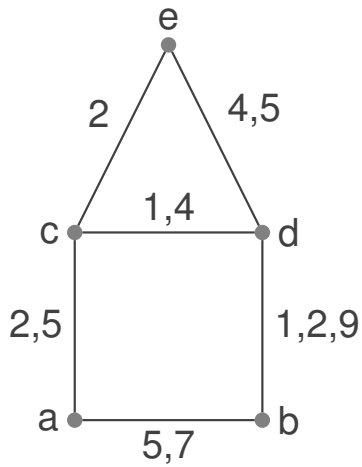

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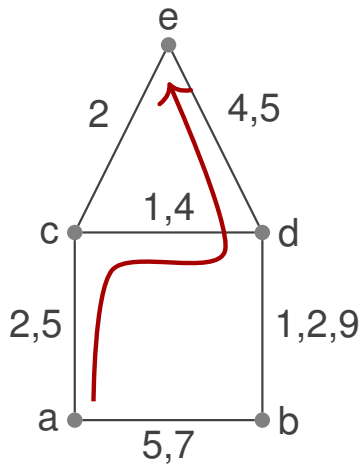

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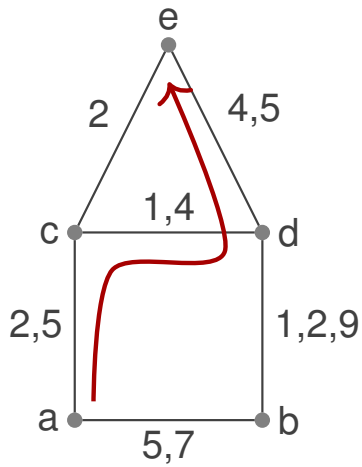

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Temporal paths

- e.g. $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ (*strict*)
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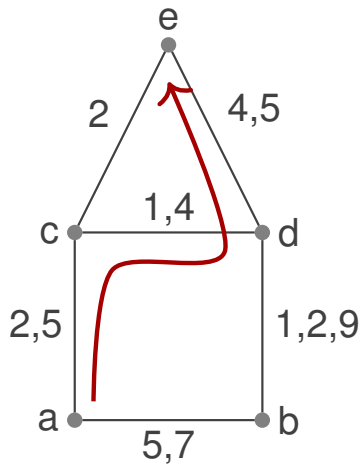

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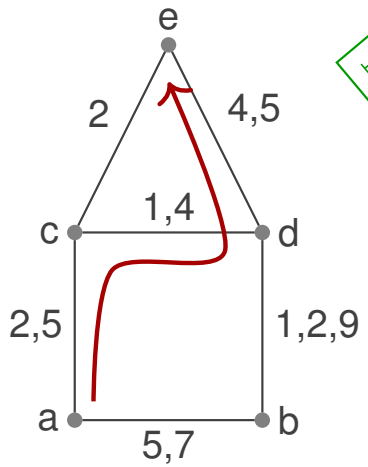

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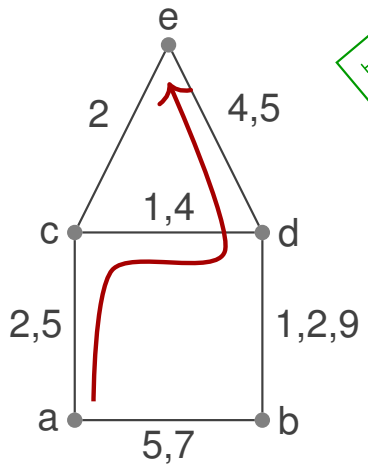

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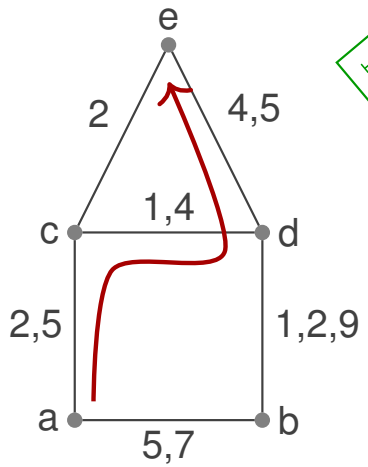

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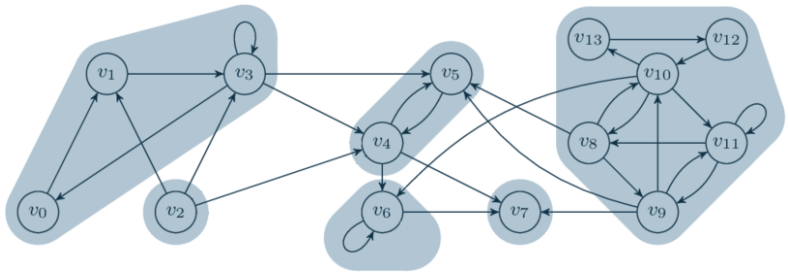
Impact of non-transitivity (Example: CONNECTED COMPONENTS)

In static graphs



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- Easy to compute

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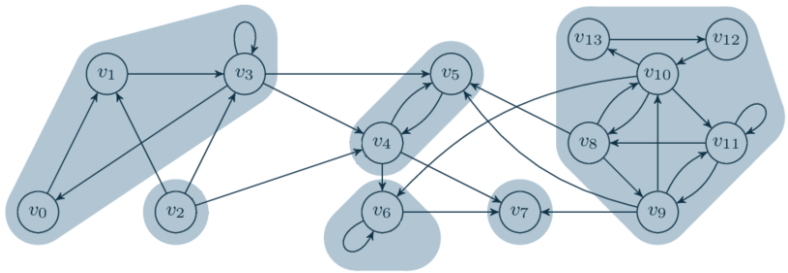

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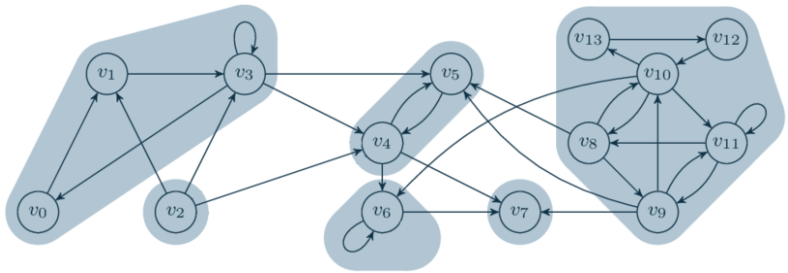

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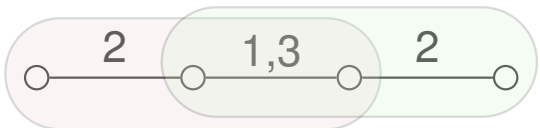

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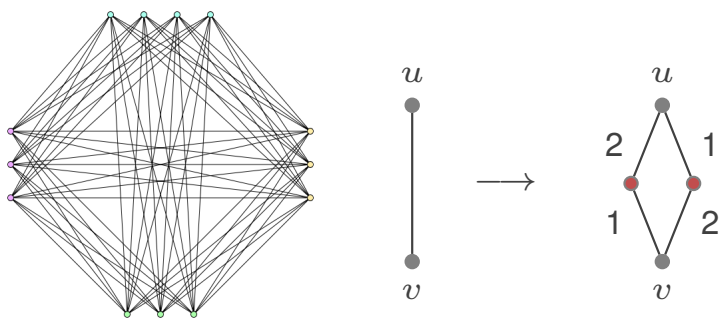
COMPONENT is NP-hard! (from CLIQUE) [Bhadra, Ferreira, 2003]



- Replace edges with semaphore gadgets
- Cliques $\Longleftrightarrow$ temporal components

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In static graphs

- Existence is guaranteed
- Size is always $n - 1$

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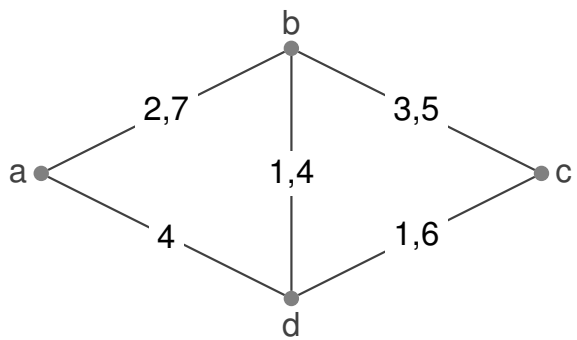
Temporal spanning tree ?

Input: A temporal graph $\mathcal{G} \in$ TC.

Goal: Find a spanning tree S of the *footprint*, so that $\mathcal{G}[S] \in$ TC.



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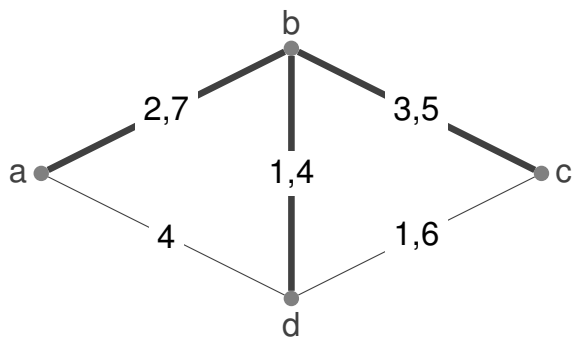

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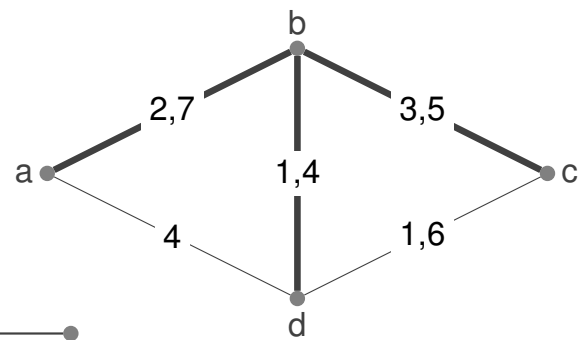

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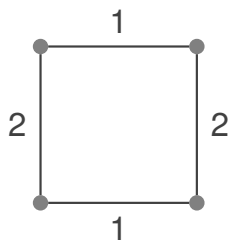

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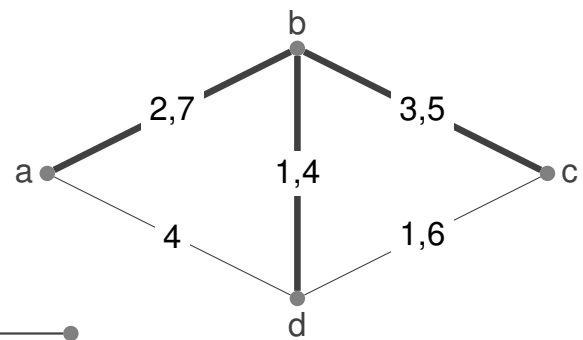

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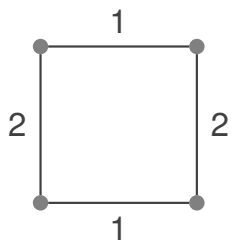
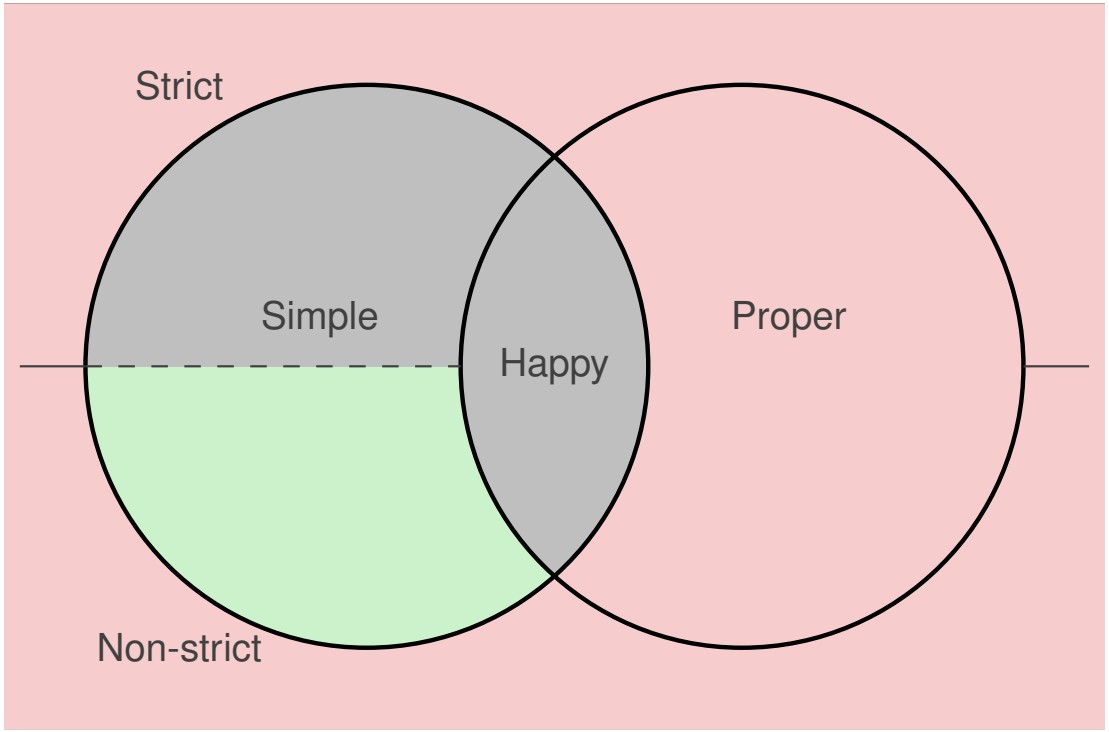


In fact, NP-hard to decide!

[C., Corsini, 2024]

Lanscape of hardness for SPANNING TREE



NP-Hard / Polynomial / Always no

Searching for the lost tree

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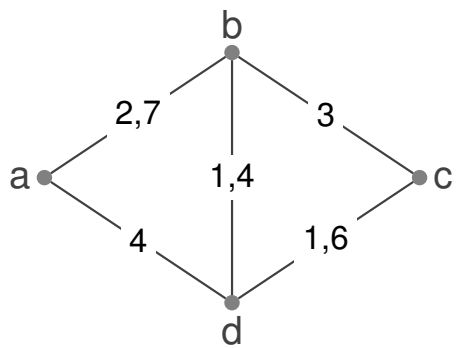

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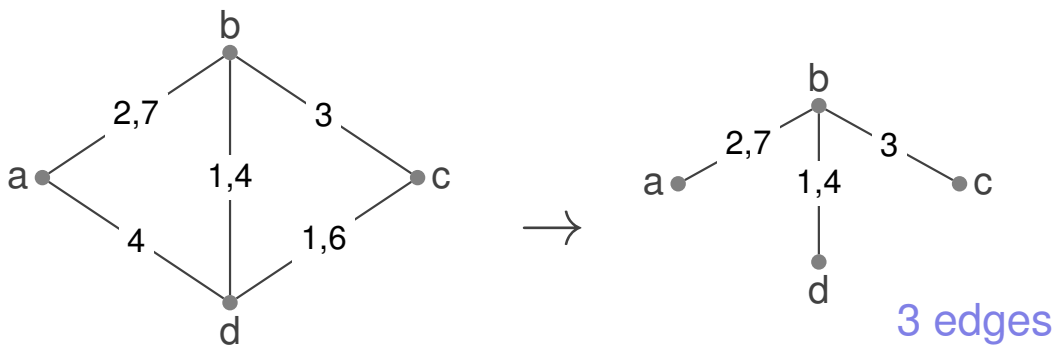

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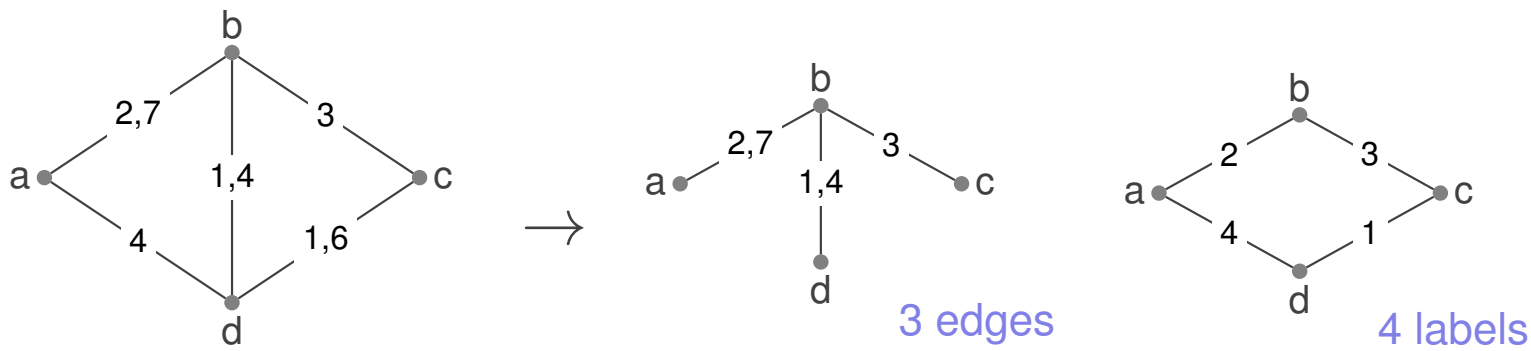

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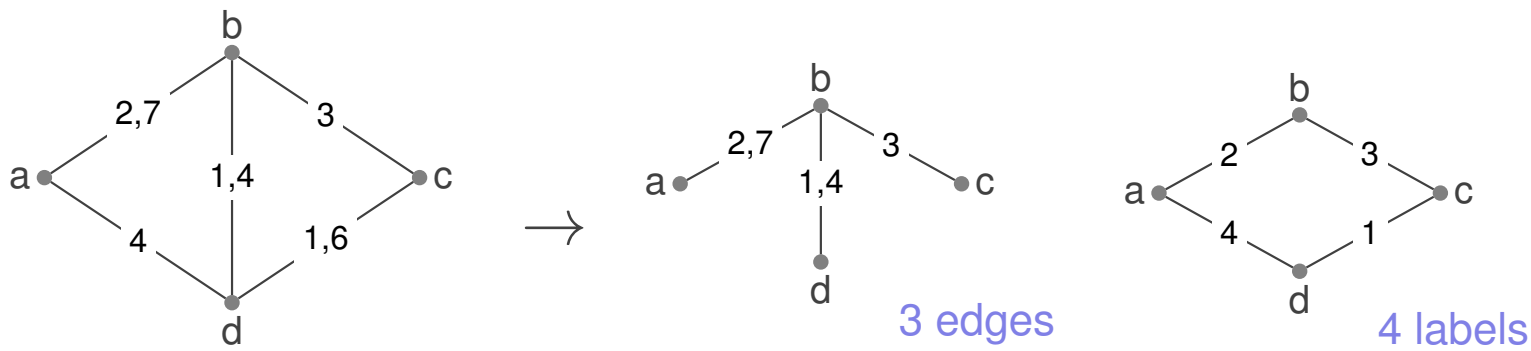

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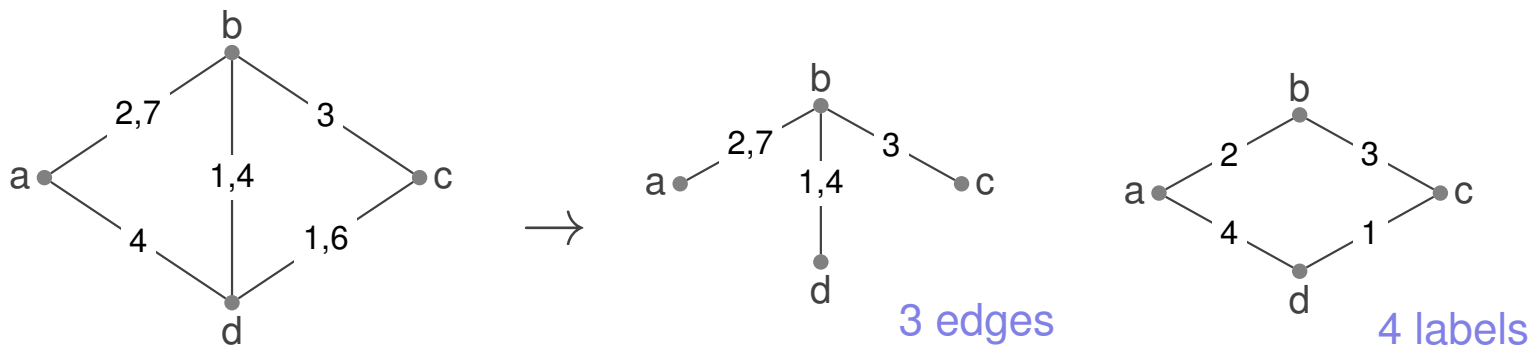

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Open for happy graphs (i.e. both simple and proper), could be polynomial.

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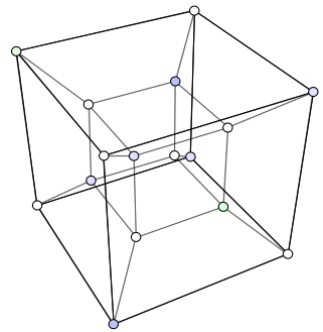

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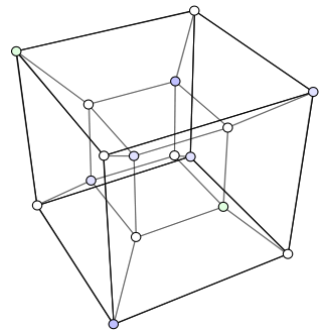

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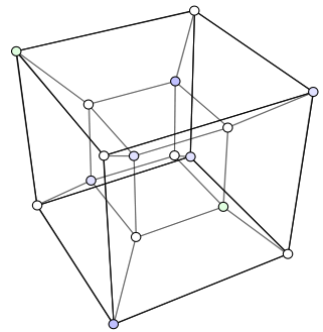

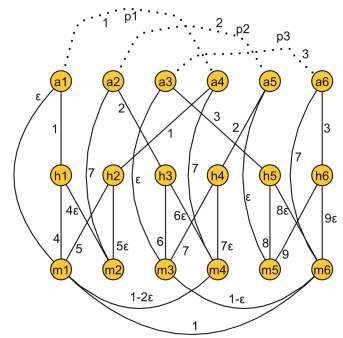

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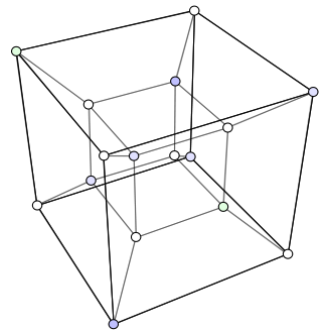

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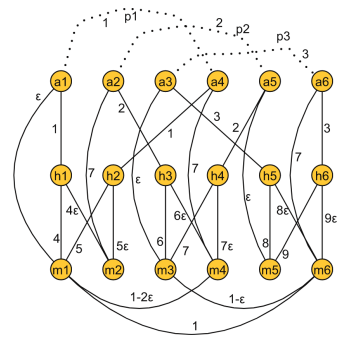

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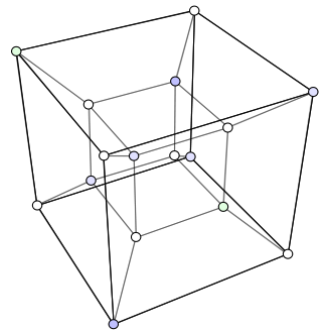

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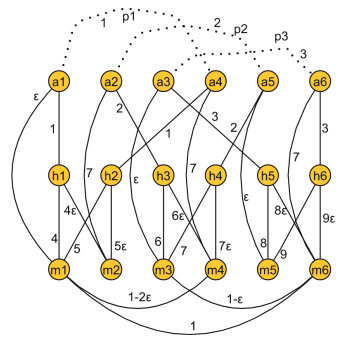

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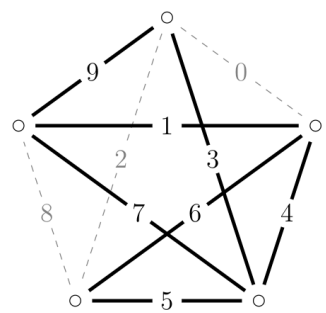

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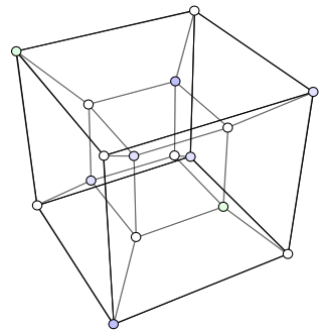

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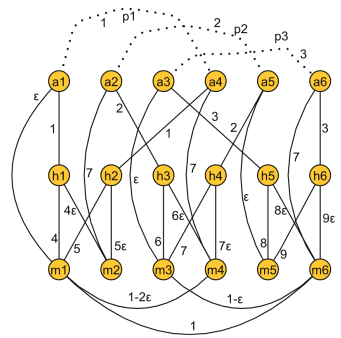

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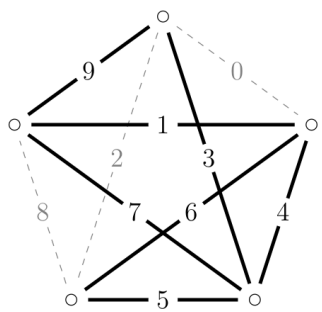

Open: $O(n)$ in temporal cliques?

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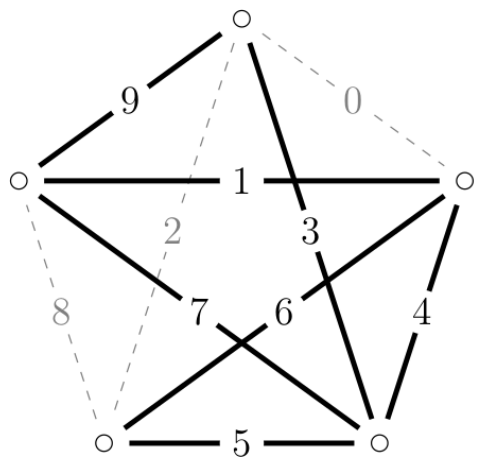

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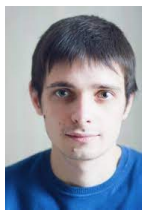

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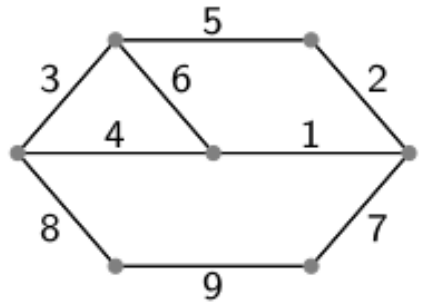

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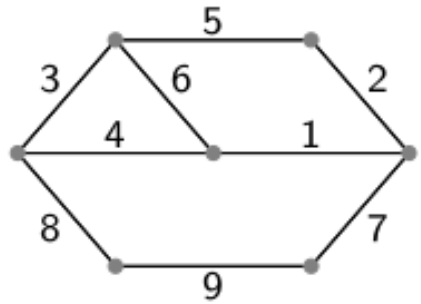

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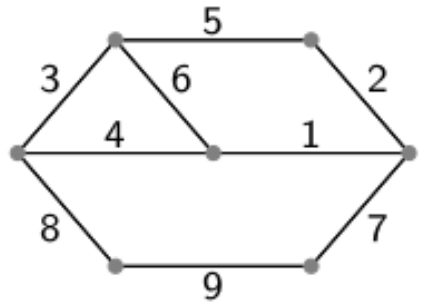

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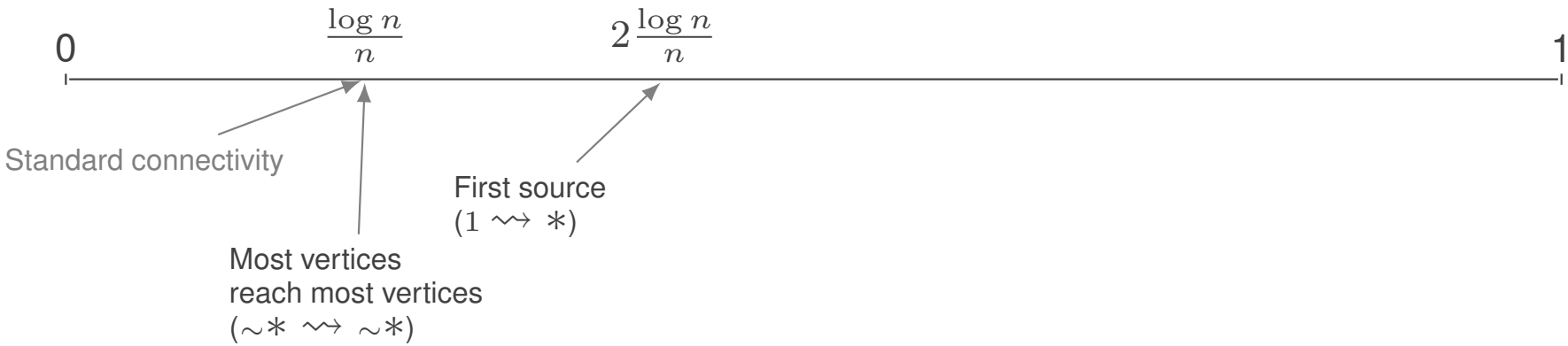

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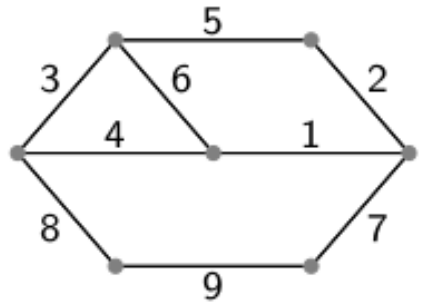

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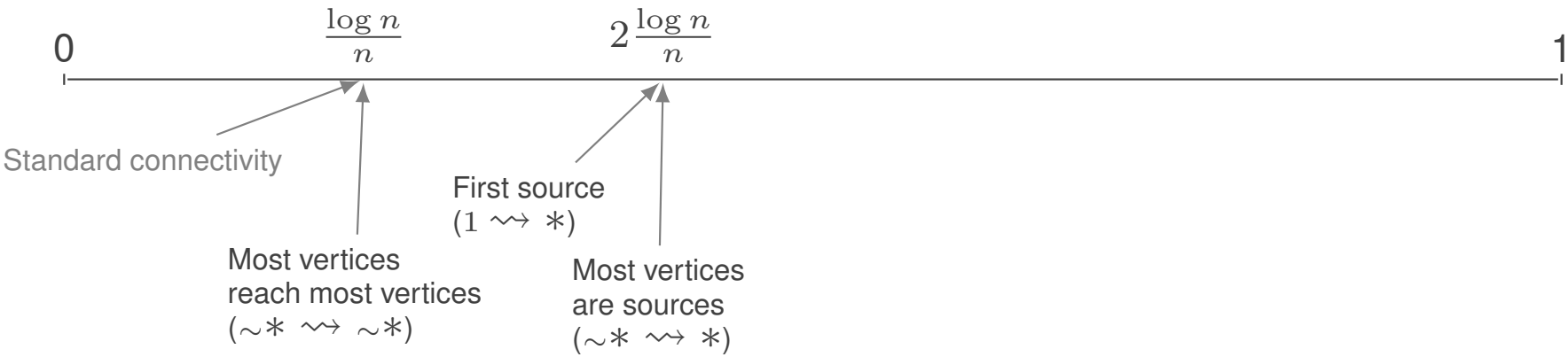

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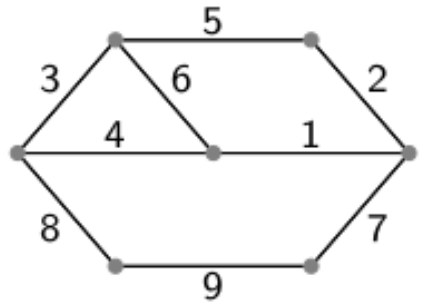

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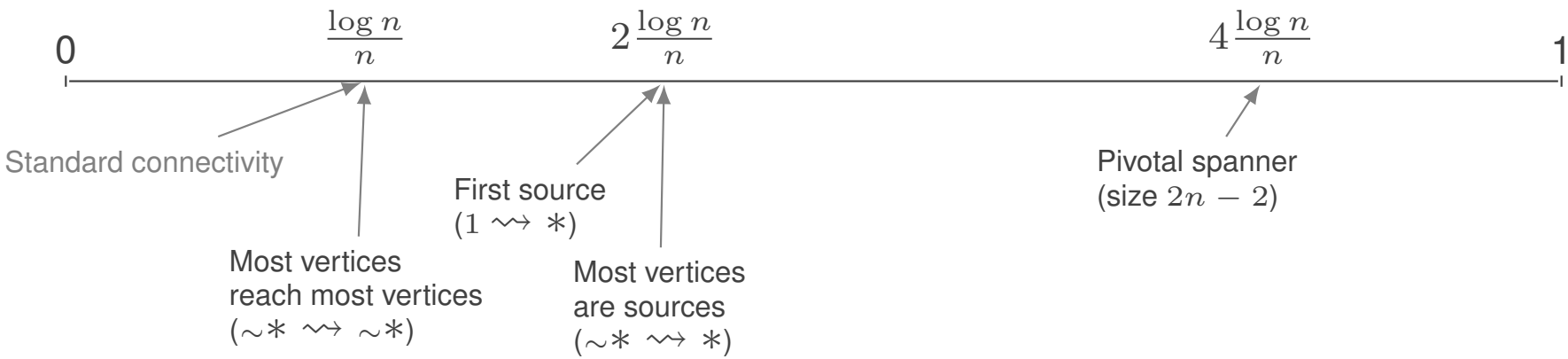

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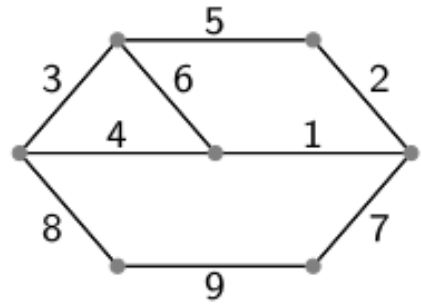

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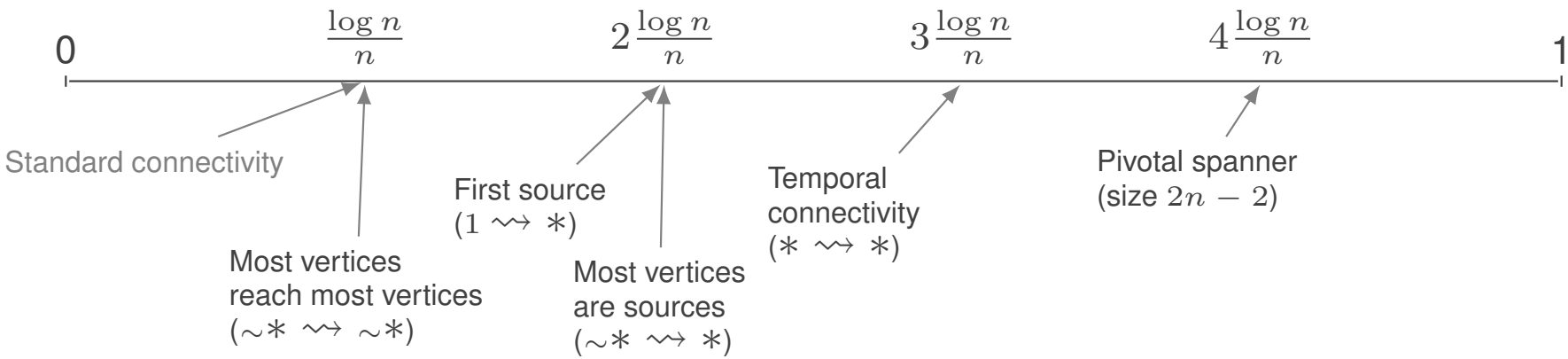

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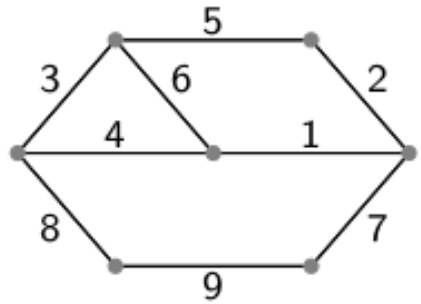

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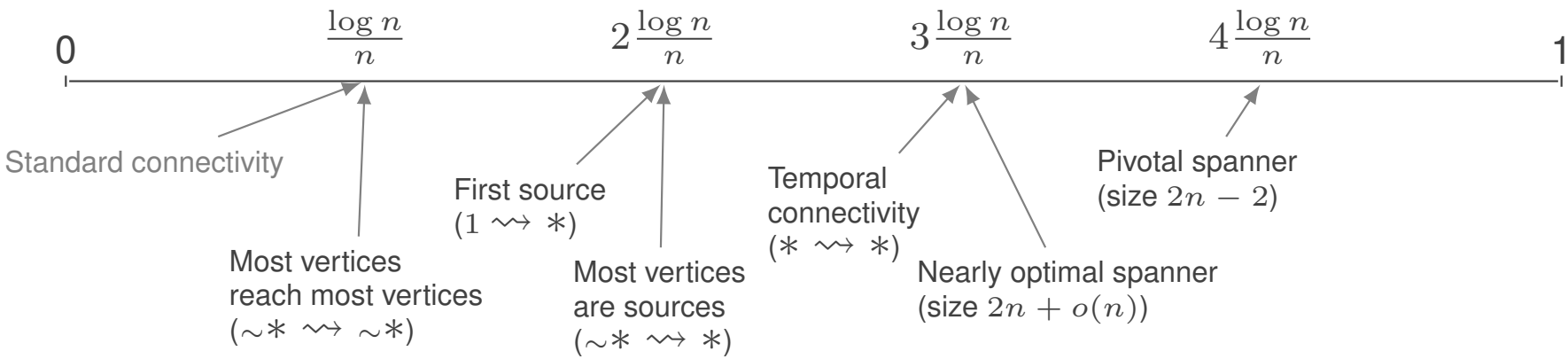

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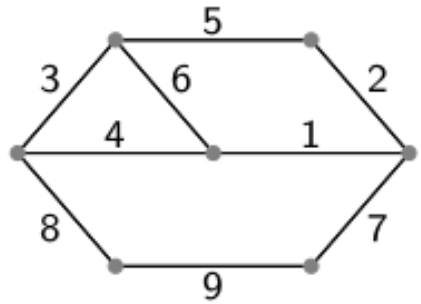

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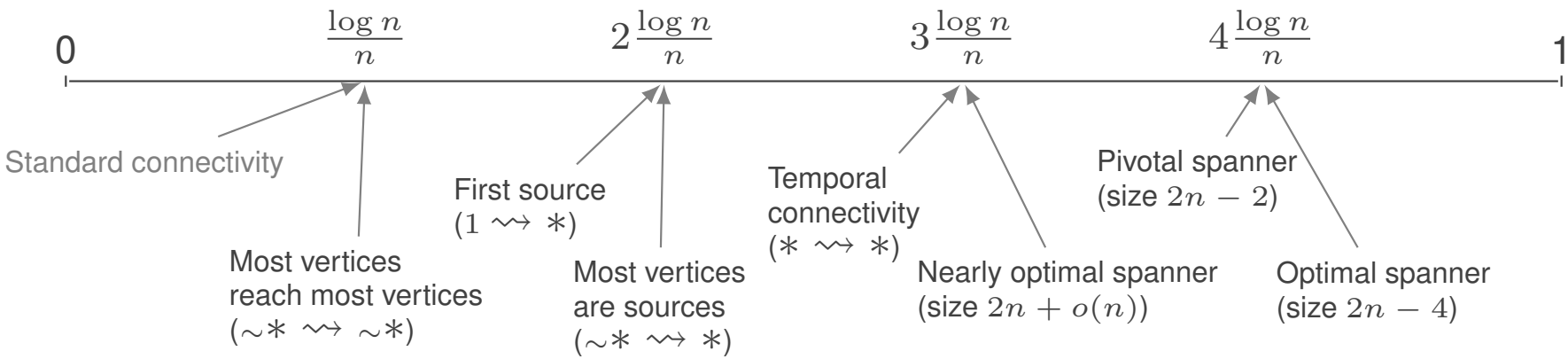

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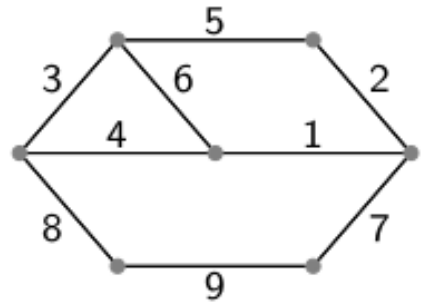

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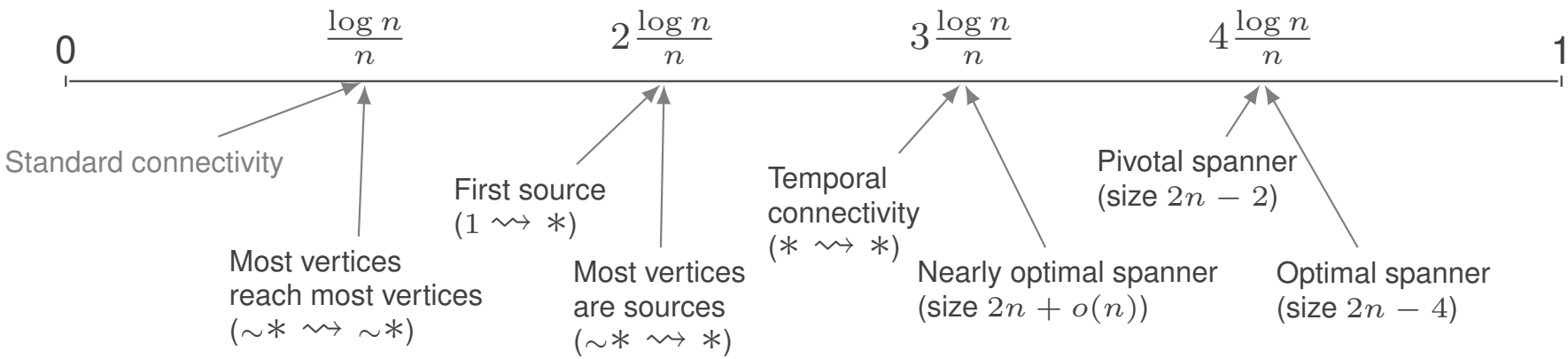
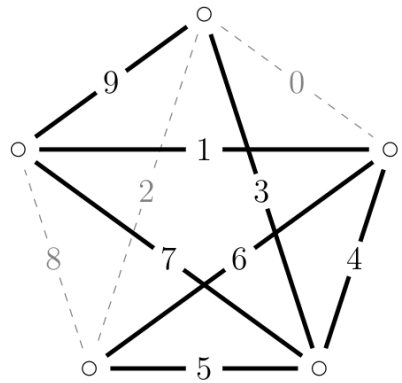


All the thresholds are sharp.

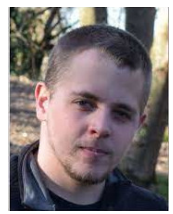
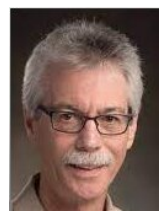
(sharp: $\exists \epsilon(n) = o(1)$, not true at $(1 - \epsilon(n))p$, true at $(1 + \epsilon(n))p$)

Good news 2:

Temporal cliques admit sparse spanners



(with)

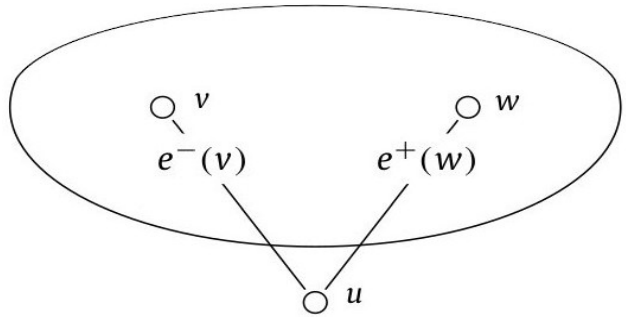
Dismountability

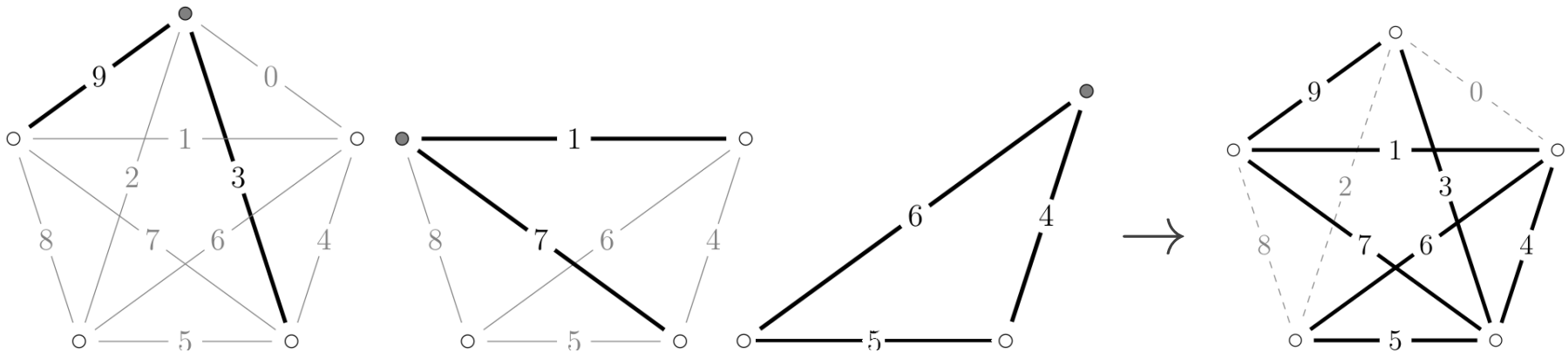
Find a node u s.t. :

- $uv =$ min edge of v
- $uw =$ max edge of w

Then `spanner(`$\mathcal{G}$`)` := `spanner(`$\mathcal{G}[V \setminus u]$`)` + uv + uw

$\rightarrow$ Recurse.





spanner of size $2n - 3$.

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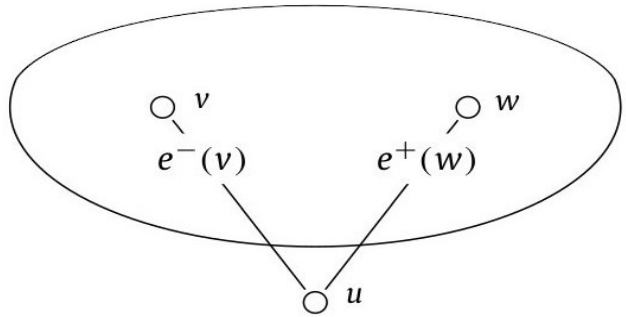

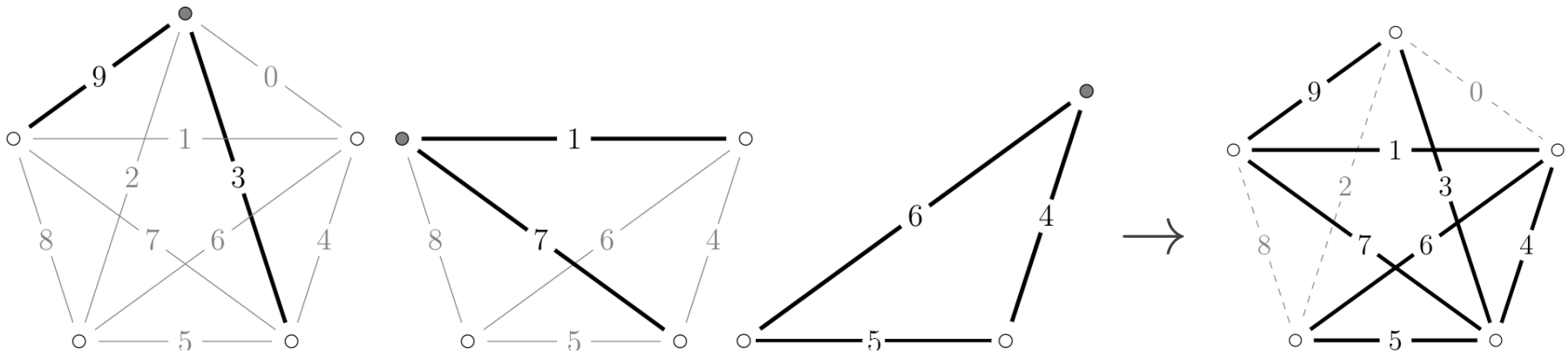

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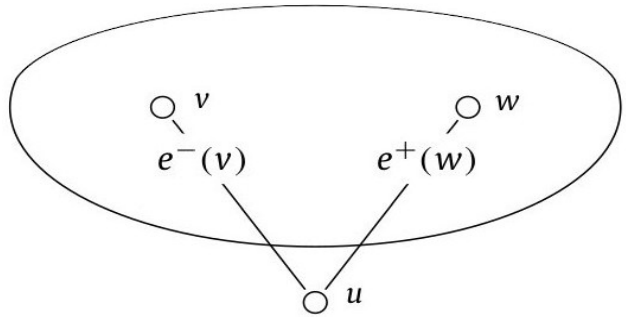

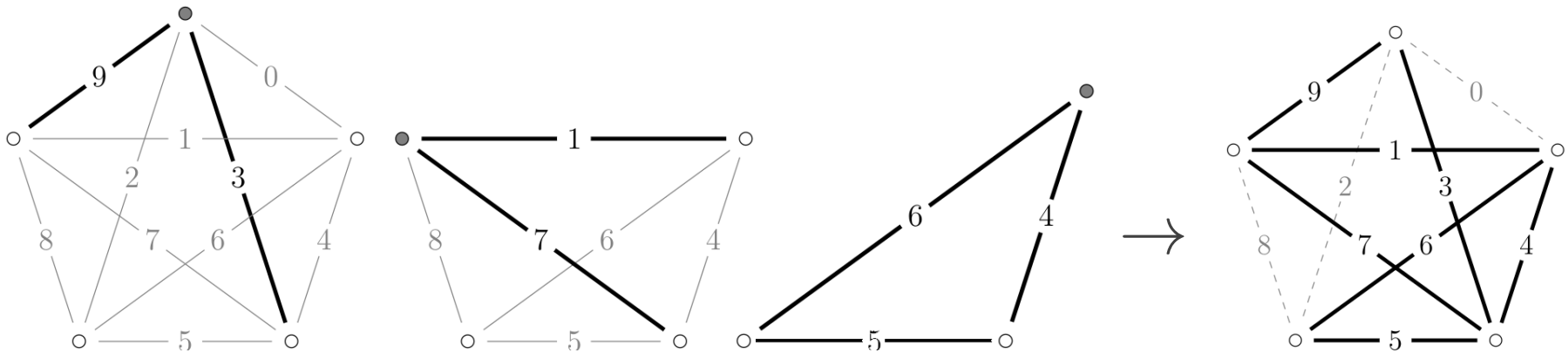

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Relaxed version: k-hop dismountability

- Temporal paths $u \rightsquigarrow v$ ending at $e^-(v)$ and $w \rightsquigarrow u$ starting at $e^+(w)$

Select these $2k$ edges, then recurse $\rightarrow$ $O(n)$-spanner if k constant.

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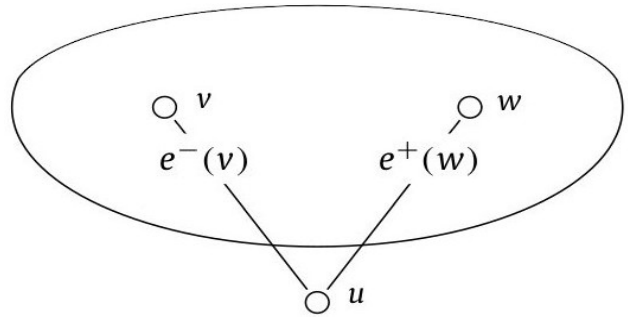

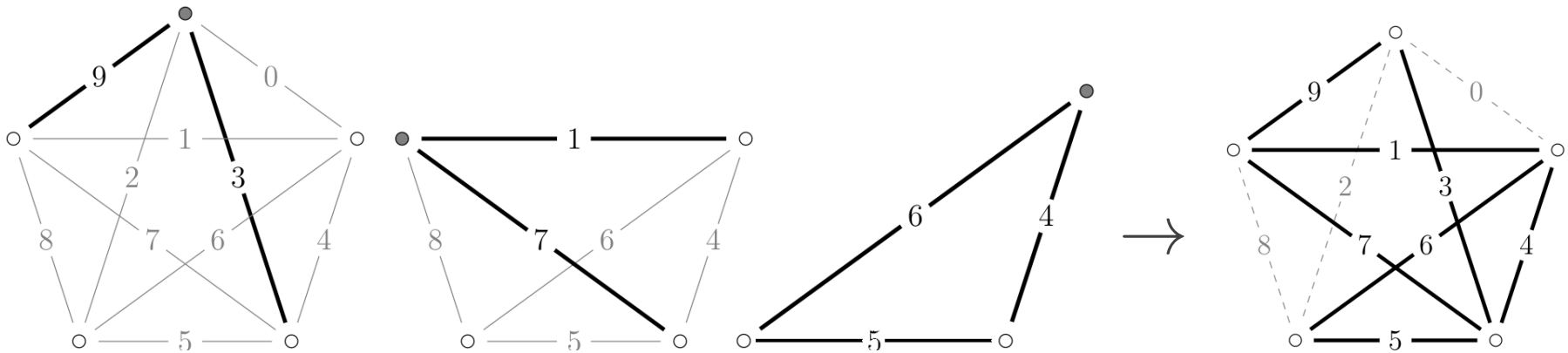

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Not always feasible, but...

What if dismountability fails?

If $\mathcal{G}$ is neither 1-hop nor 2-hop dismountable, then the following is guaranteed:

- Complete bipartite graph $\mathcal{H} \subseteq \mathcal{G}$ ($n/2$ vertices in each part)
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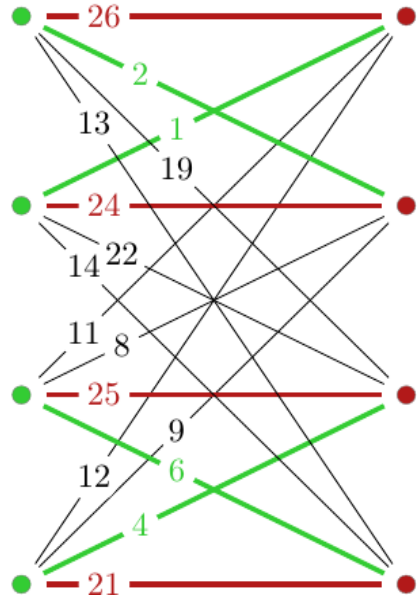
- **A spanner of $\mathcal{H}$ is a spanner of $\mathcal{G}$**



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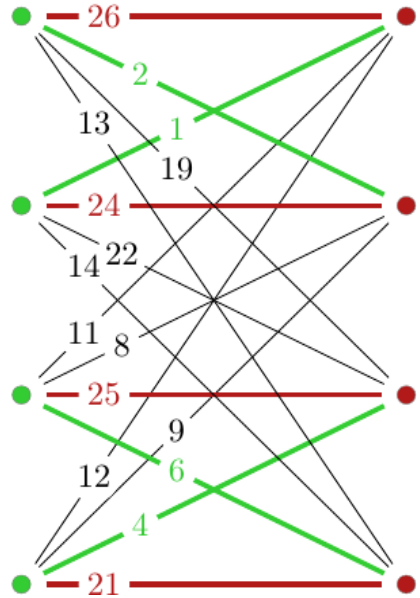

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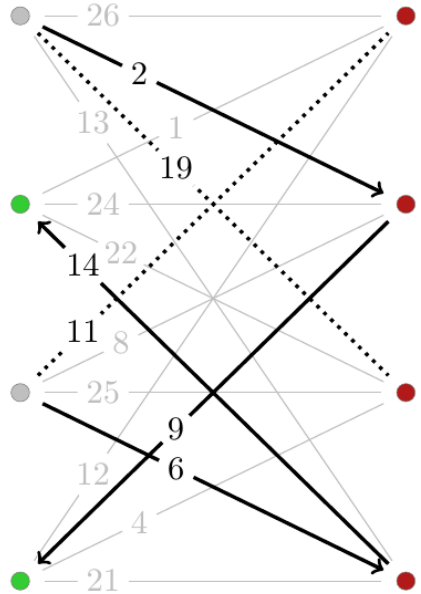

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Half of the green nodes can be iteratively removed, at doubling cost.
Repeat $\log n$ times.

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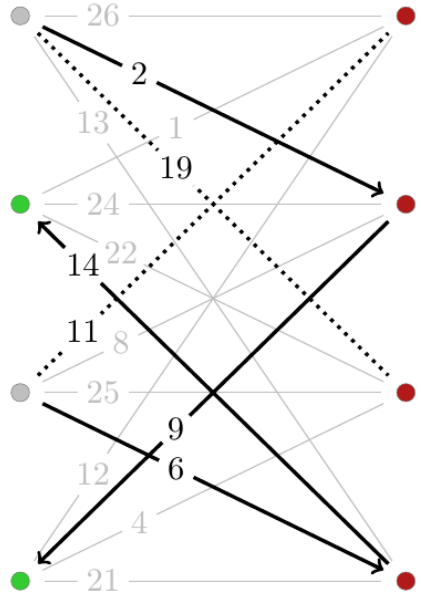

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$\rightarrow$ Spanners of size $O(n \log n)$ always exist.

Summary and open questions

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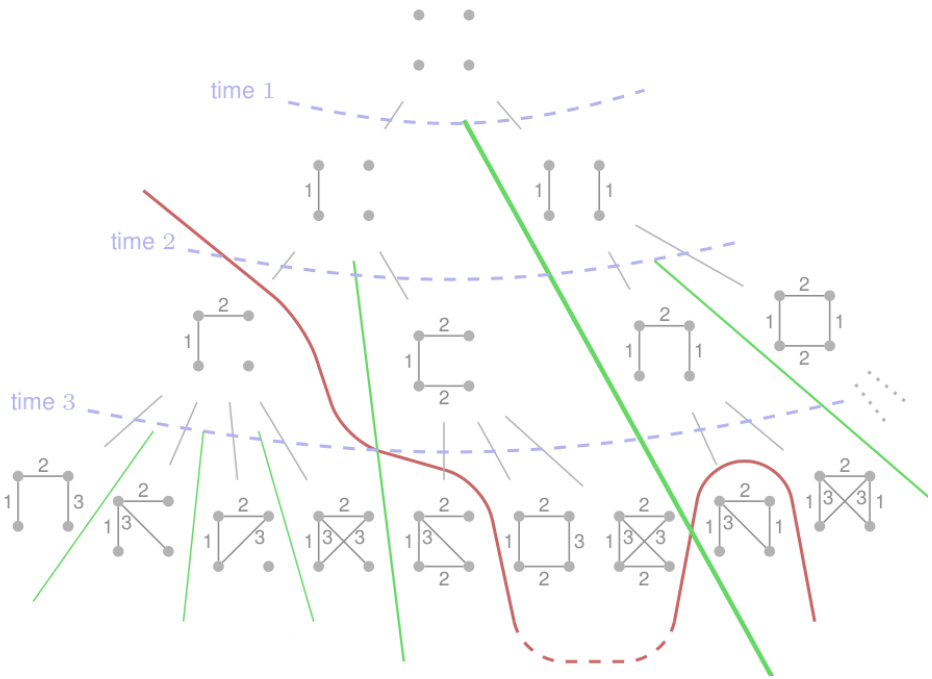
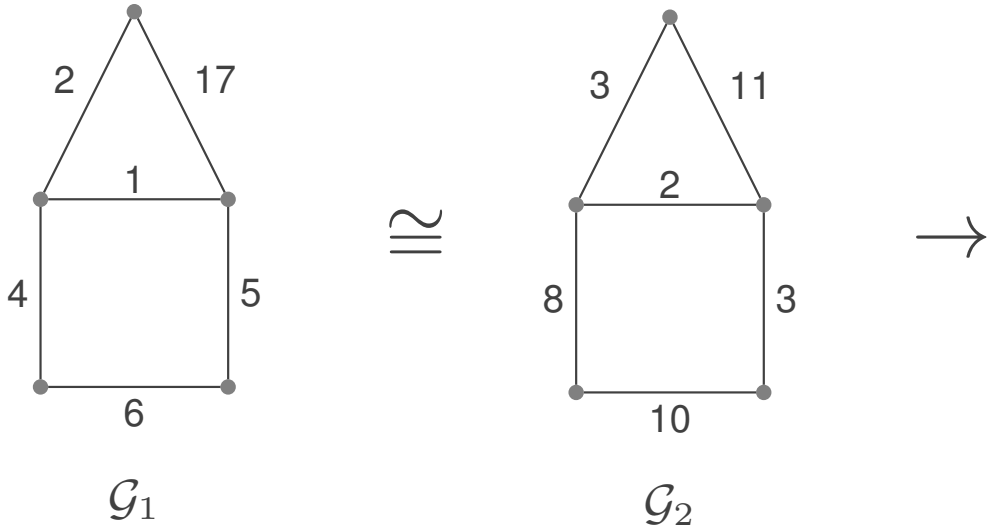
Enumeration of happy temporal graphs



Equivalence based on reachability



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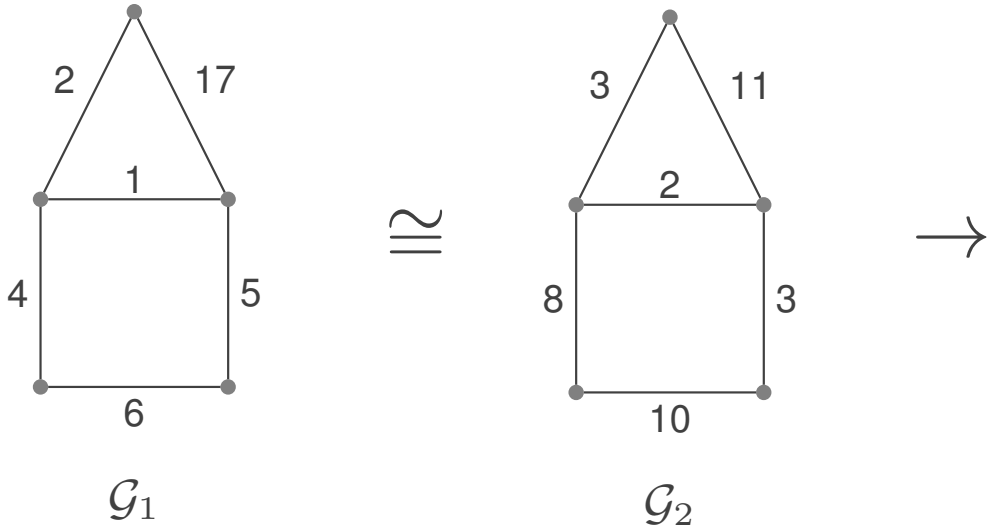

How to capture this equivalence?

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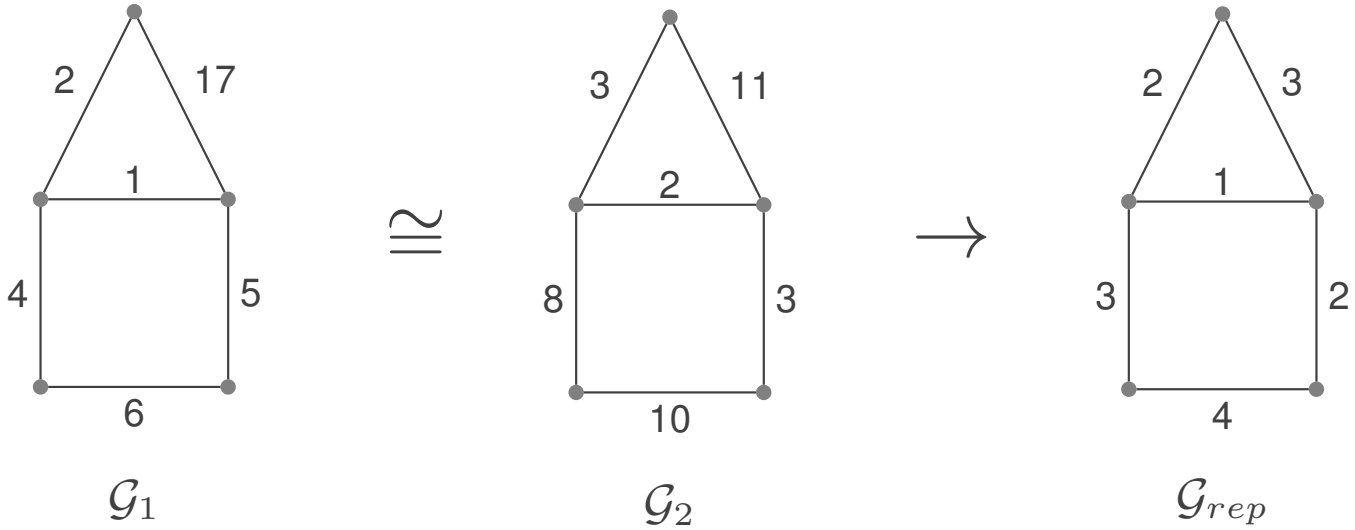
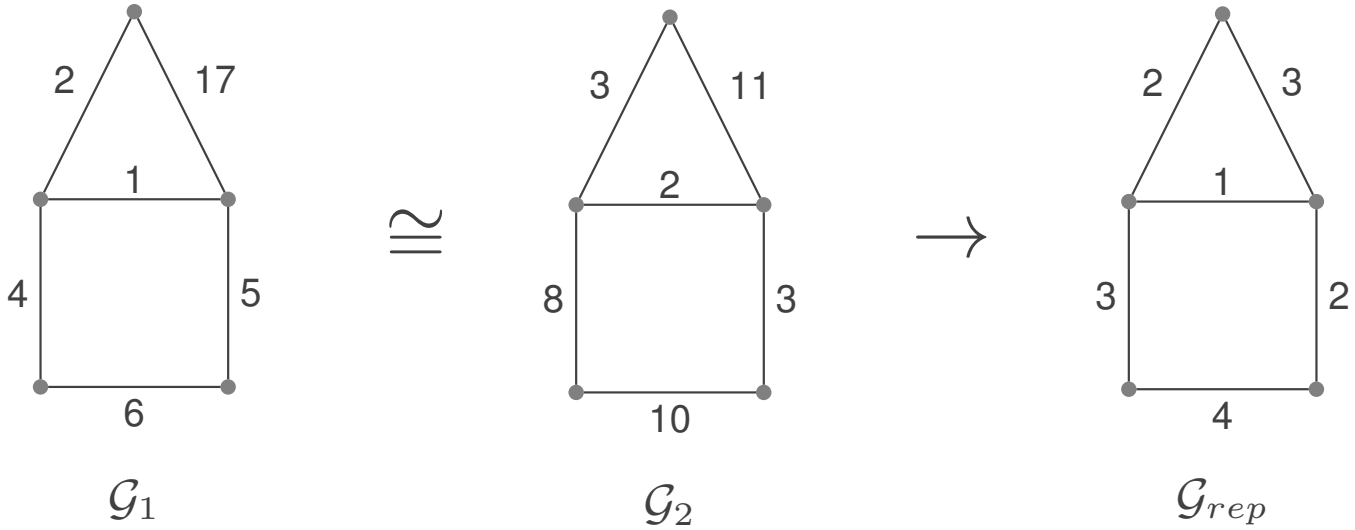


How to capture this equivalence?

- Canonical representatives

Equivalence based on reachability



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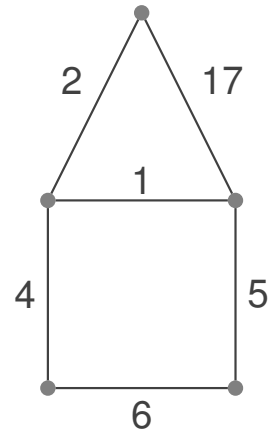
Good news:

- Finite number
- Canonization, isomorphism testing, and generators of the automorphism group all computable in *polynomial time*.

Properties of representatives

Canonization

1. Find minimal edges
2. Give smallest available time
3. Repeat



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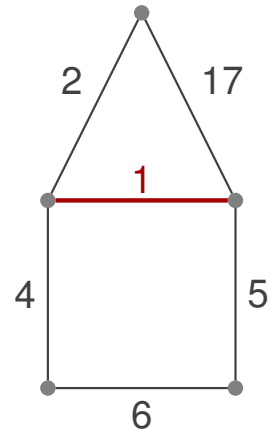

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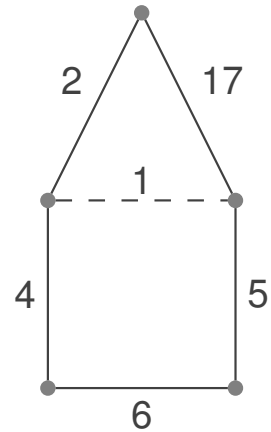

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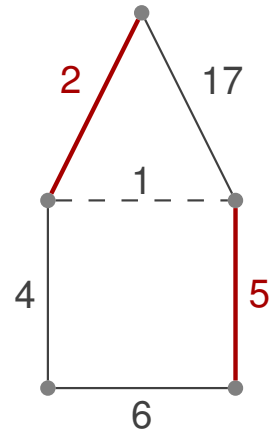

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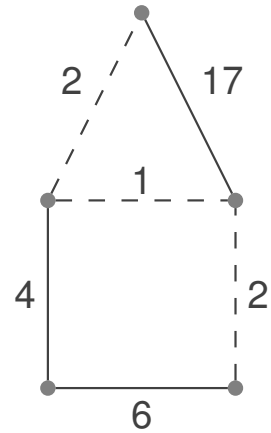

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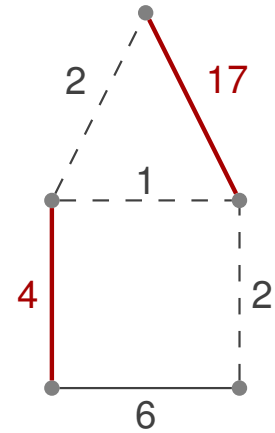

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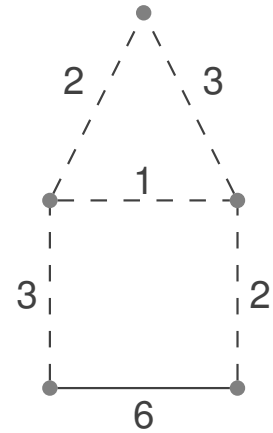

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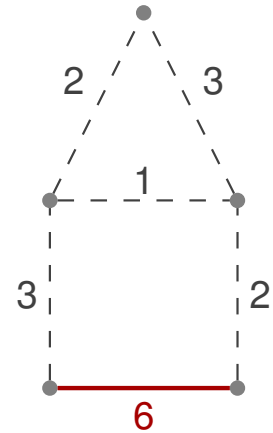

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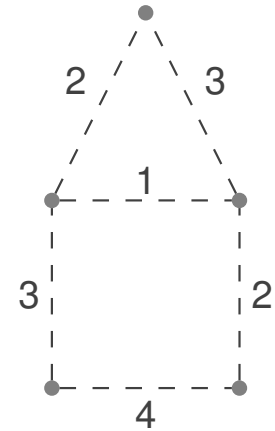

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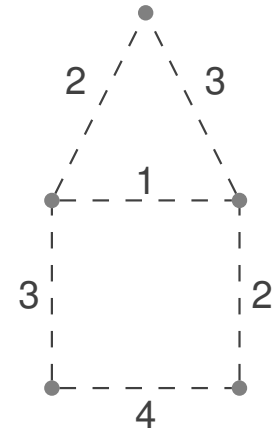

Contiguity Lemma: If an edge is labeled $t > 1$, then an adjacent edge is labeled $t - 1$.

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Input: Two happy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$

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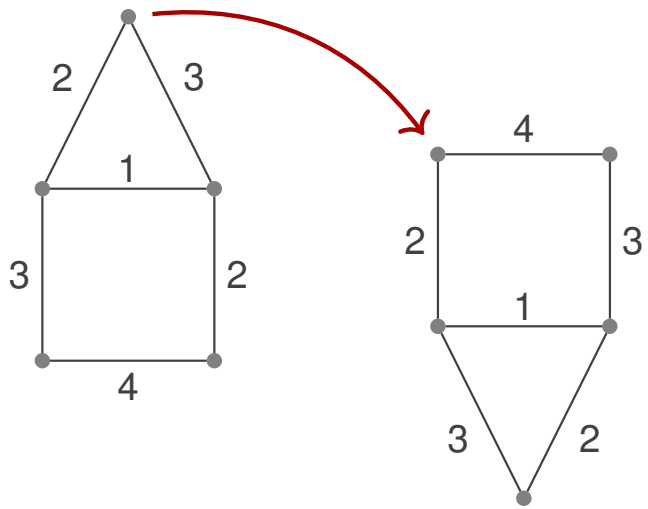

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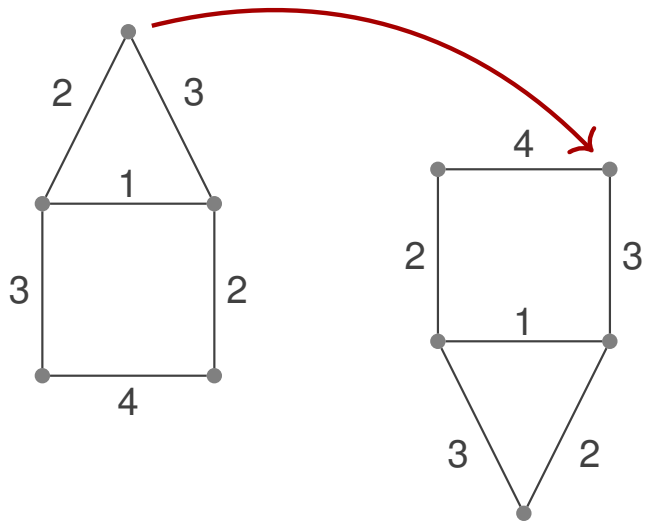

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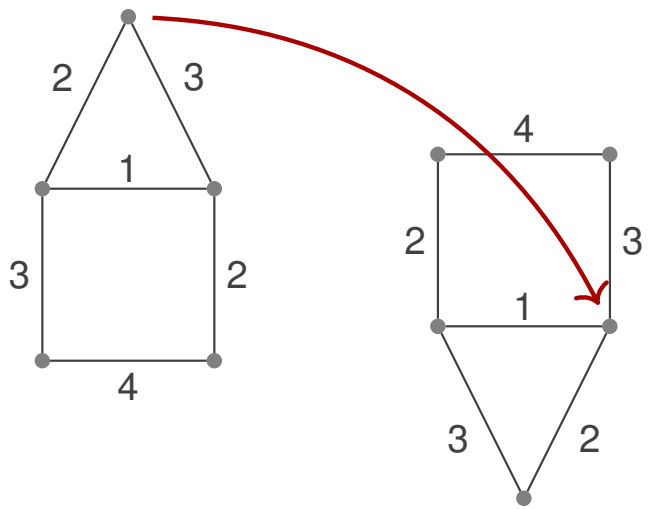

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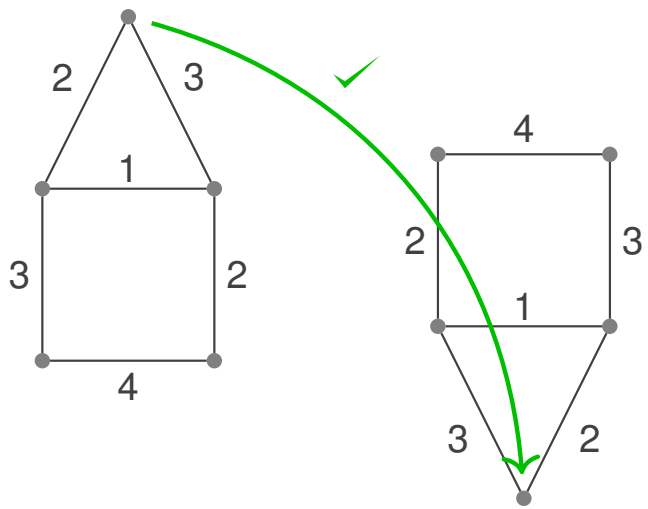

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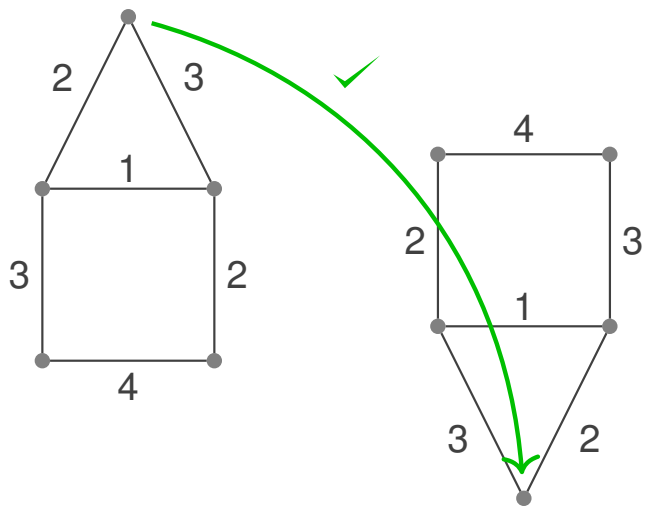

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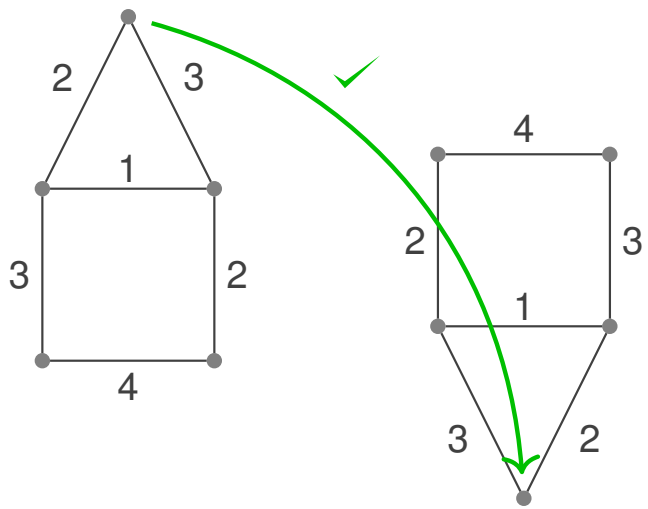

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Remark: Also feasible using Babai & Luks machinery (1983)

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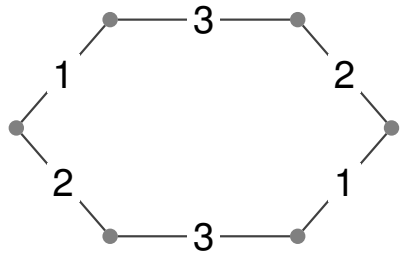
$\rightarrow$ Same strategy as for isomorphism.



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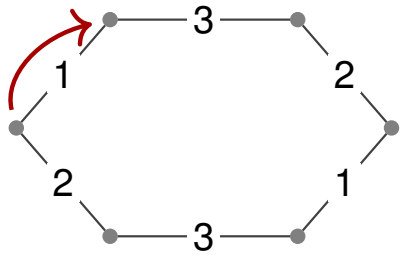

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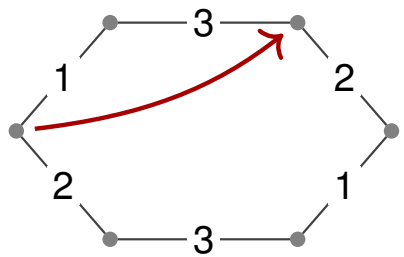

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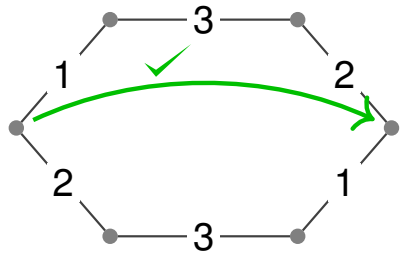

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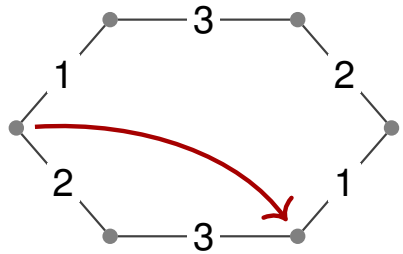

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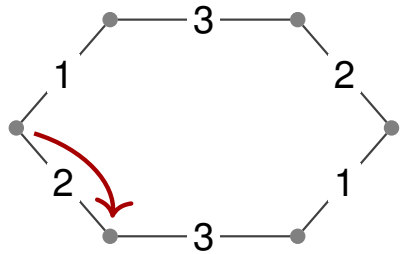

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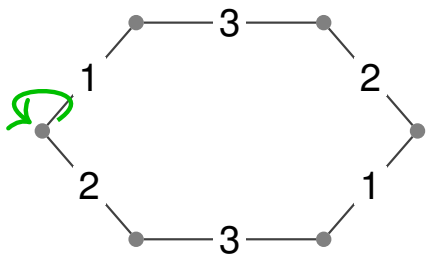

At most n automorphisms!

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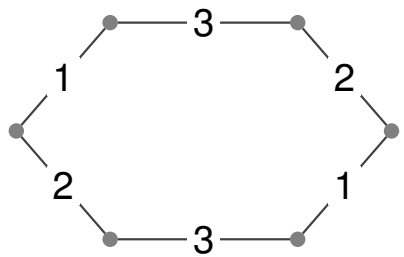

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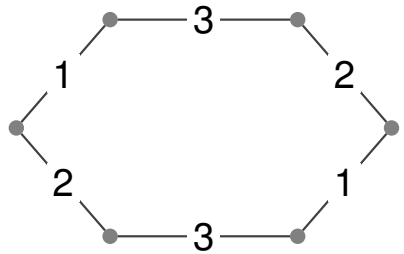

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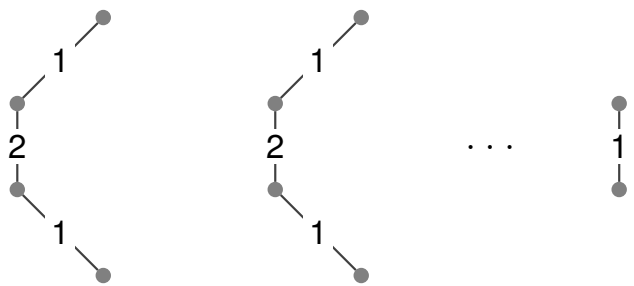

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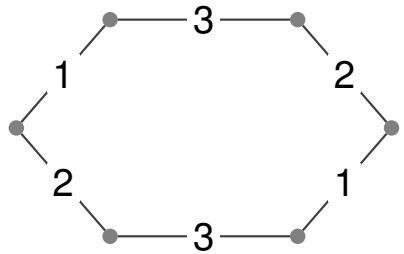

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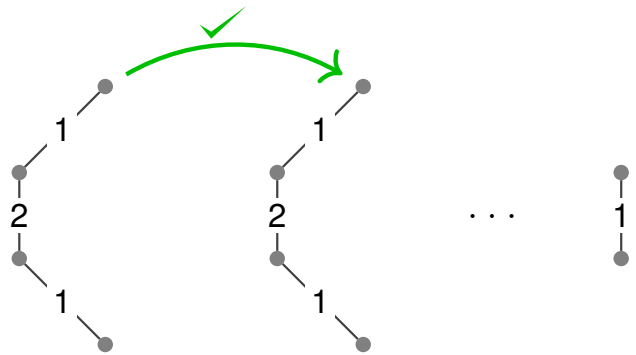

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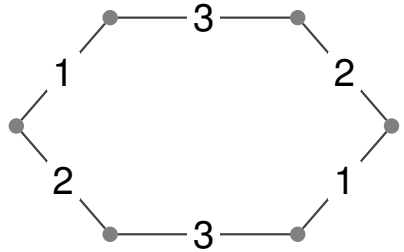

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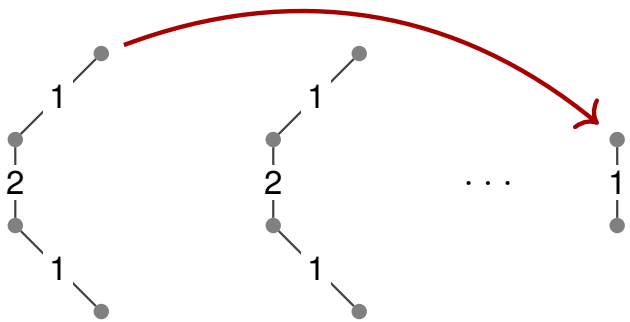

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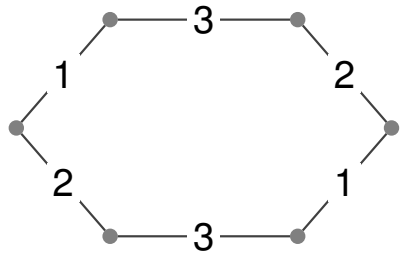

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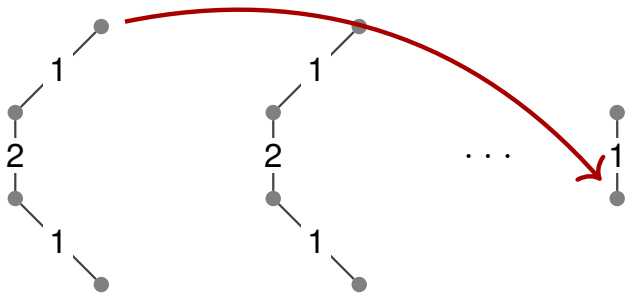

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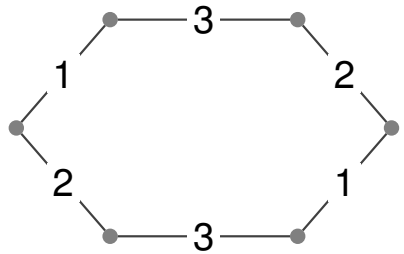

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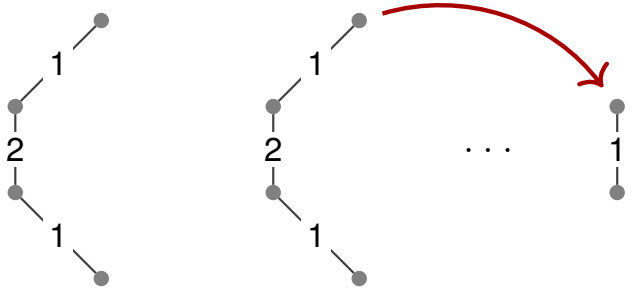

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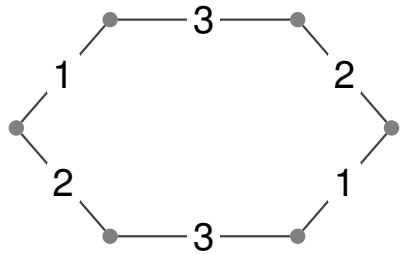

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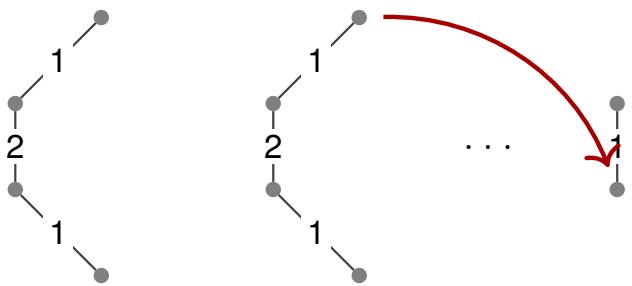

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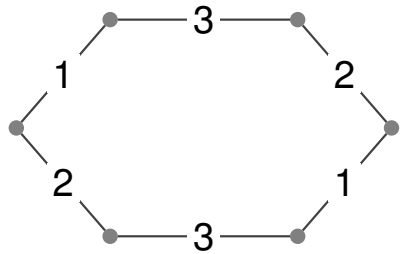

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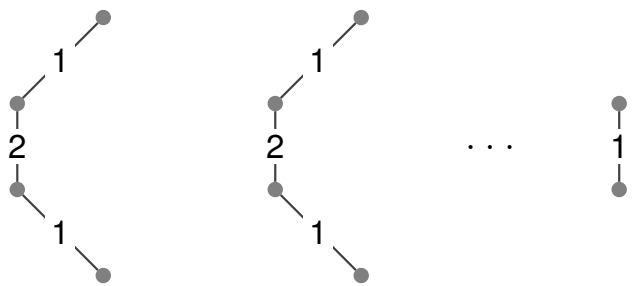

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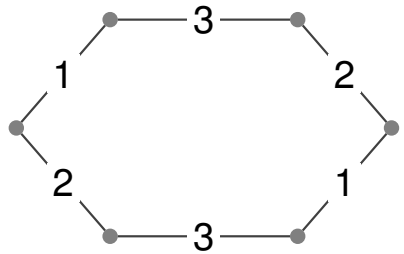

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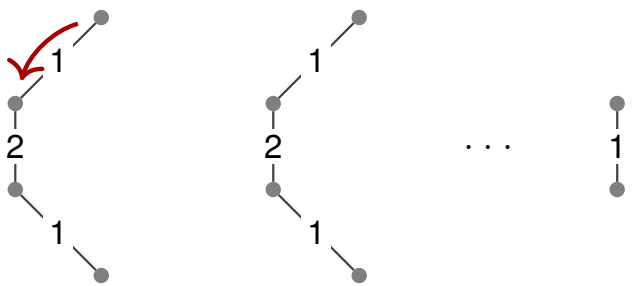

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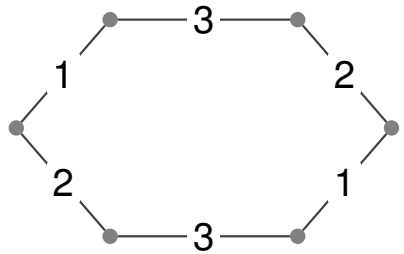

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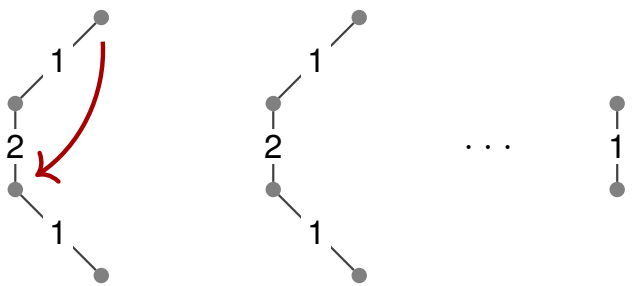

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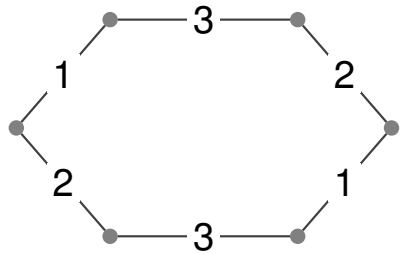

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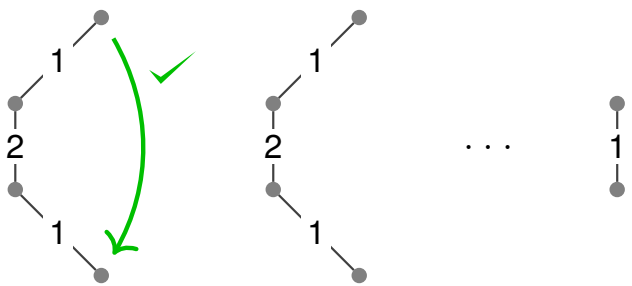

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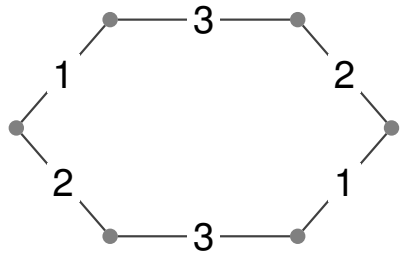

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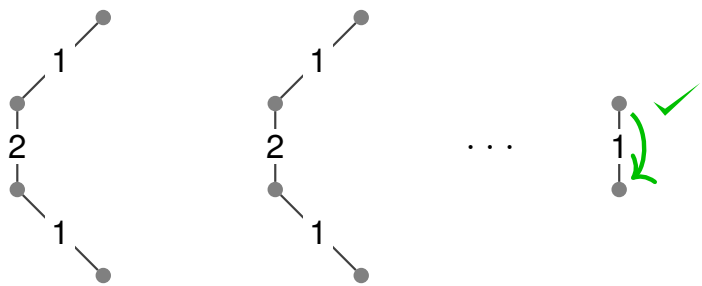

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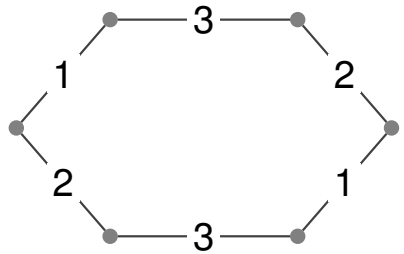

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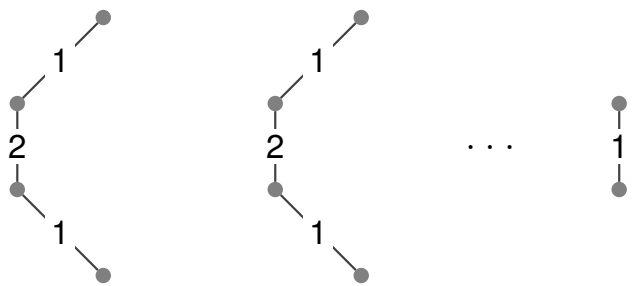

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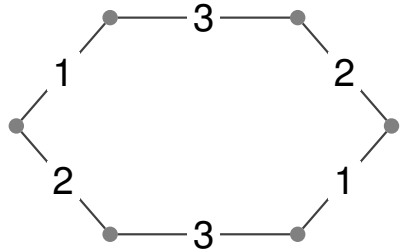

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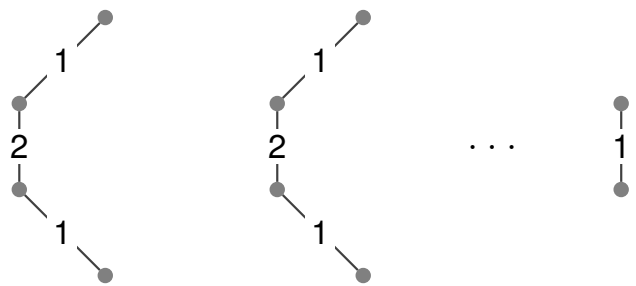

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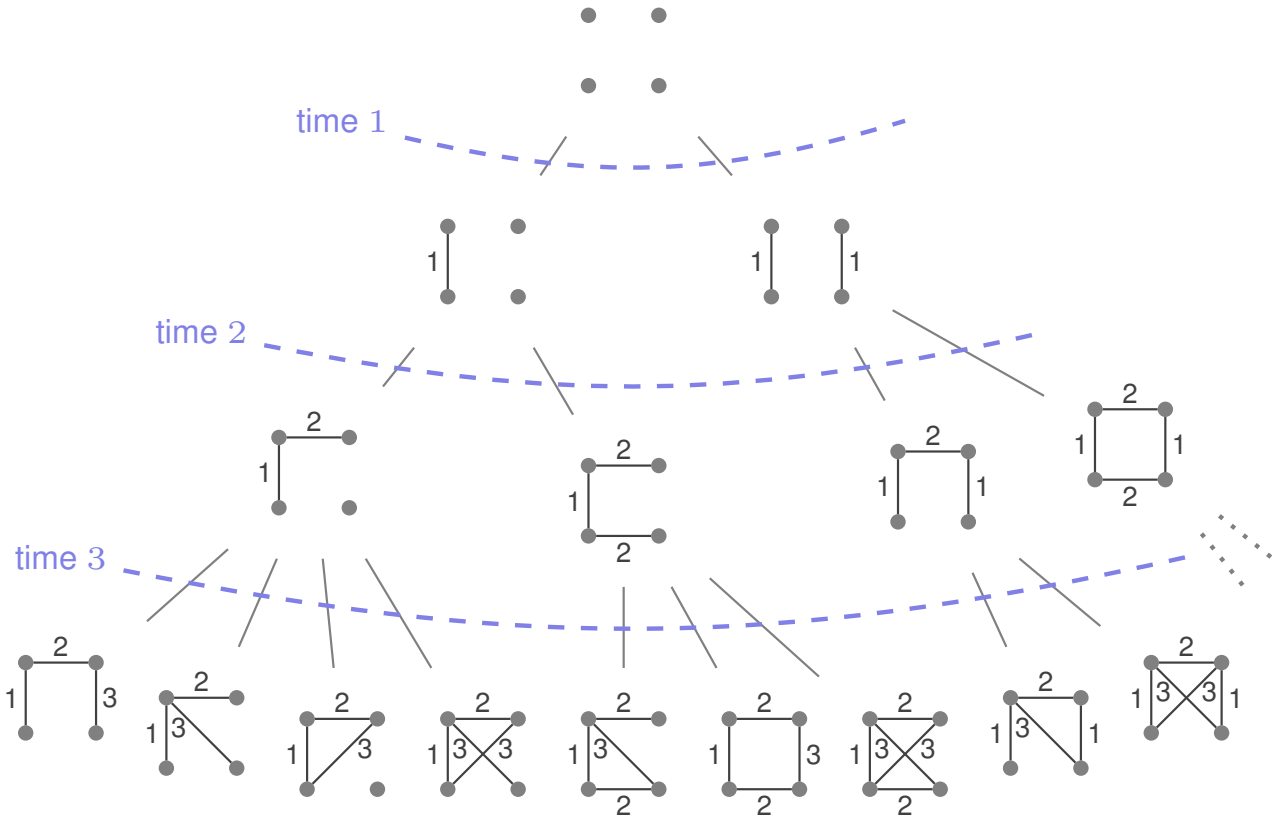
$\rightarrow$ *Generators in polynomial time!*

Generation tree

Principle: One level = one time

$\rightarrow$ successors $=$ all possible ways to add the *next time*



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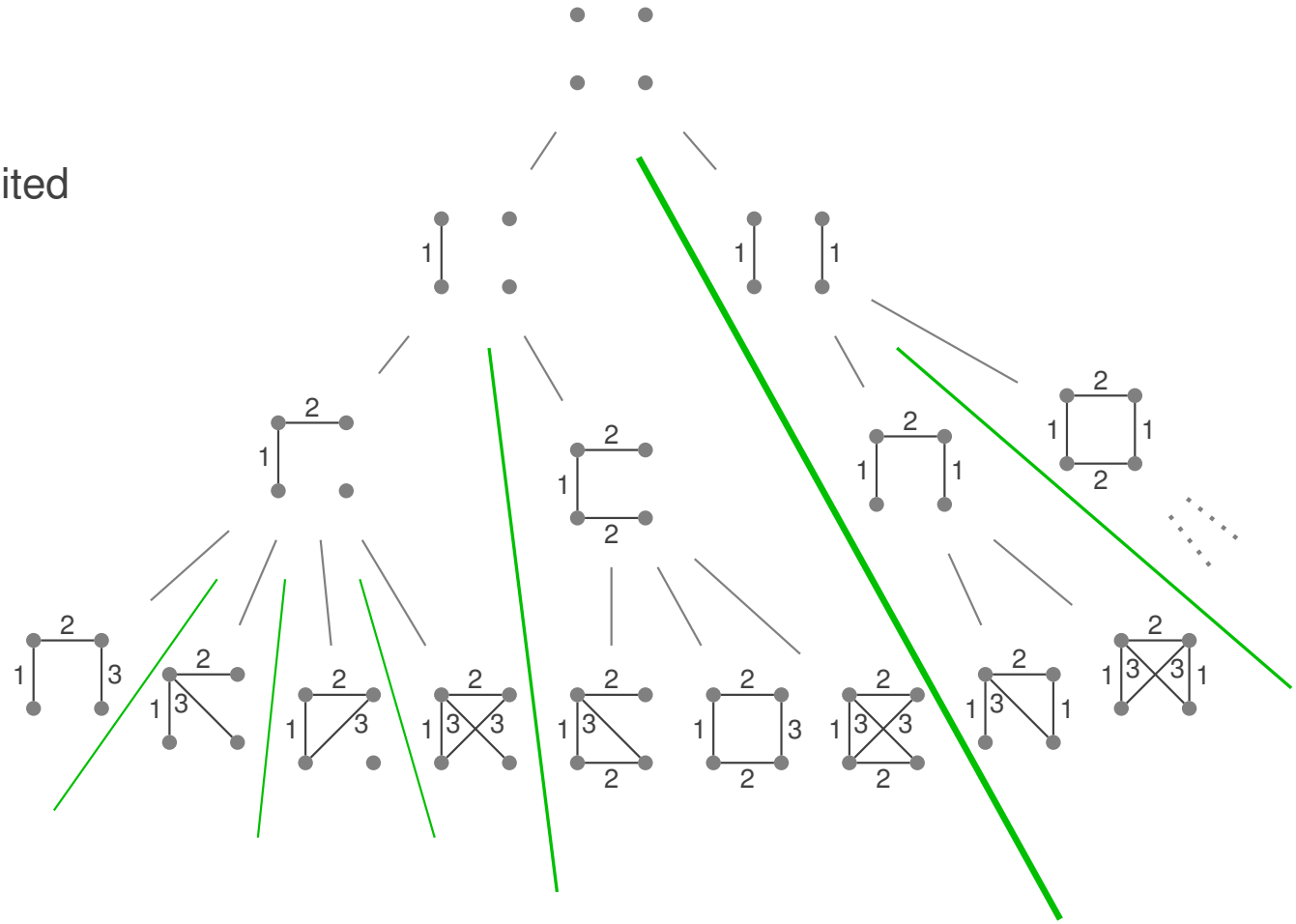
Key properties

1. Dissimilarity is inherited



$\downarrow$ Isomorphism types separated (forever)

Generation tree

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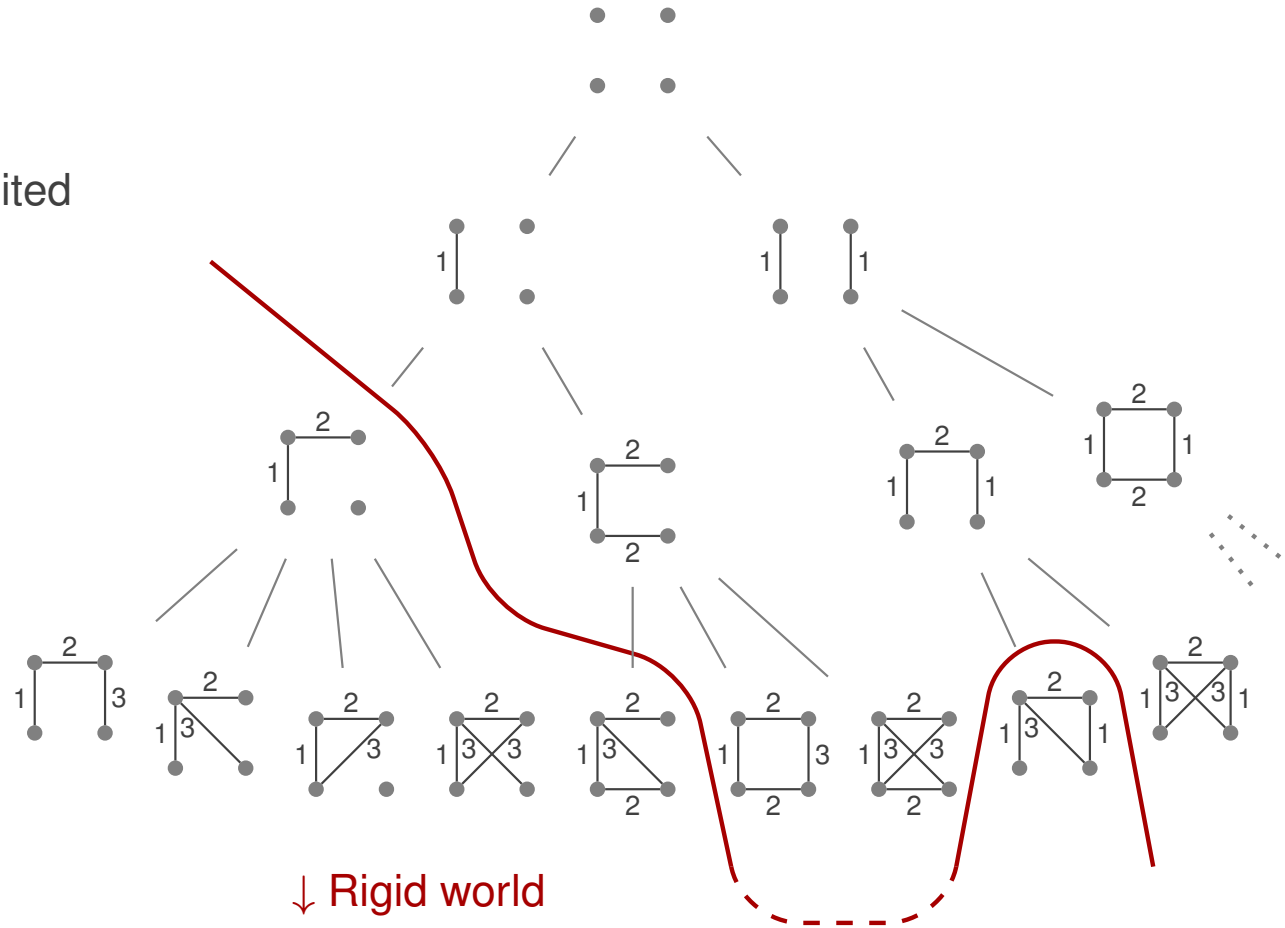
Key properties

1. Dissimilarity is inherited
2. Rigidity is inherited



How to compute successors?

Input: A happy representative $\mathcal{G}$, max time t

Output: Happy representatives extending $\mathcal{G}$ with time $t + 1$.

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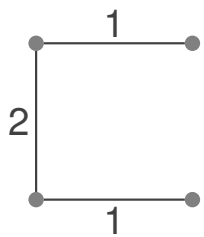

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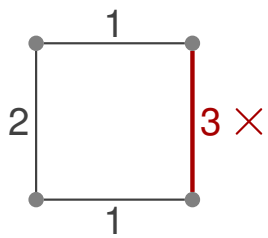

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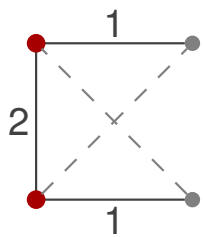

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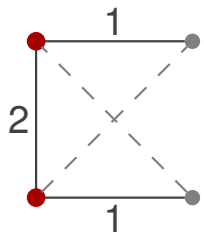

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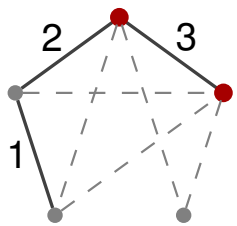
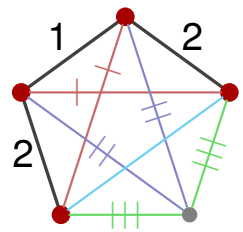
$\mathcal{G}$ is rigid



Two cases

$\mathcal{G}$ has symmetries



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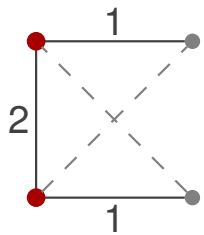

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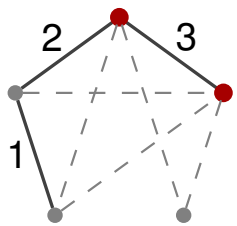

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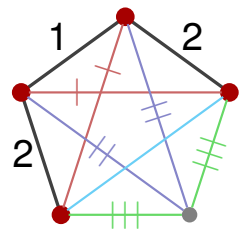

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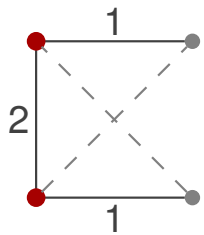

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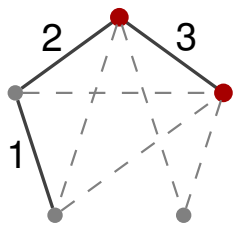

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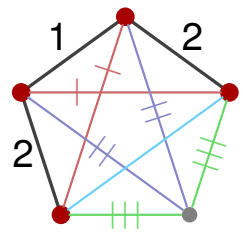
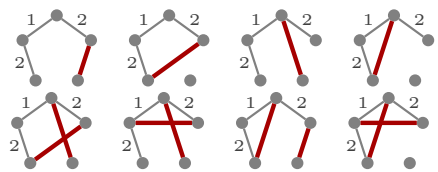


→ Enumerate matchings of eligible *non-edges* whose *multisets of orbits* are distinct



Back to the 2n-3 conjecture

Various implementations of the generator (`Python`, `Java`, `Julia`, `Rust`, **C++**)
`https://arnaudcasteigts.net/blog/stgen.html`

Do spanners of size $2n - 3$ always exist in happy cliques?

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$\rightarrow$ True for $n \leq 8$.

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Do spanners of size $2n - 3$ always exist in happy cliques?

$\rightarrow$ True for $n \leq 8$.

Some numbers

n	# Happy graphs (representatives)	# Happy cliques (representatives)
3	4	1
4	62	20
5	15378	4524
6	89769096	23218501
7	13725757879376	3106952711040

Thanks!

(Battelle, Dec 3, 2023)