Temporal graph theory: paradigm and algorithmic challenges

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August 21, 2025

SympGraph
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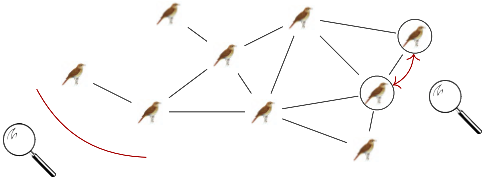
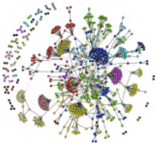
Studying networks



Network as input

→ centralized algorithms...



Network as environment

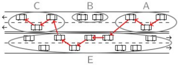
→ distributed algorithms...

The world is dynamic...

In technologies

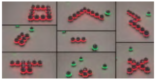






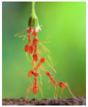




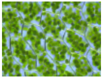
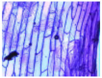

In nature





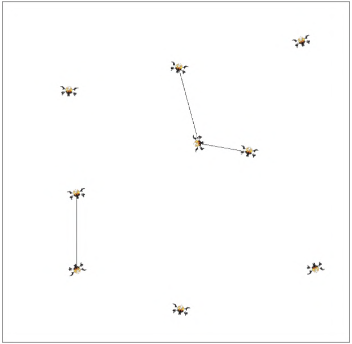





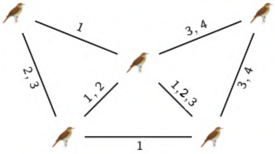




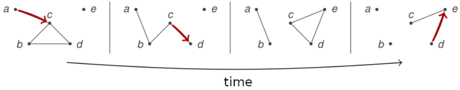

(Highly) dynamic networks?

Example of scenario



Modeling





Properties:

- Temporal connectivity? $\mathcal{TC}$
- Repeatedly? $\mathcal{TC}^{\mathcal{R}}$
- Recurrent links? $\mathcal{E}^{\mathcal{R}}$
- In bounded time? $\mathcal{E}^{\mathcal{B}}$
- ...

$\longrightarrow$ Classes of temporal graphs

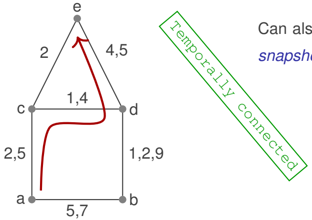
Temporal graphs

(*a.k.a.* time-varying, time-dependent, evolving, dynamic,...)

$\mathcal{G} = (\underline{V, E}, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns *time labels* to edges.

footprint of $\mathcal{G}$

Example:



Can also be viewed as a sequence of *snapshots* $\{G_i = \{e \in E : i \in \lambda(e)\}\}$

Restrictions on labeling: *simple* ($\lambda : E \to \mathbb{N}$); *proper* ($\lambda$ locally injective), *happy* (both).

Temporal paths

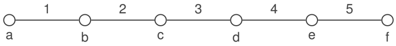
- e.g. $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ (*strict*)
- e.g. $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ (*non-strict*)

Temporal connectivity: Temporal paths between all pairs (class TC).

$\rightarrow$ Warning: In general, reachability is non-symmetrical... and non-transitive!

Time versus structure (basic observations)

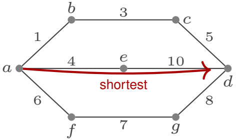
Centrality?



(time $\mod 7$)

Optimal paths?

[Bui-Xuan, Ferreira, Jarry, 2003]

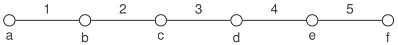


Which path is optimal from a to d?

-min hop?

Time versus structure (basic observations)

Centrality?



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Optimal paths?

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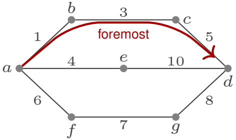


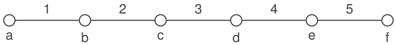
Which path is optimal from a to d?

-min hop?

-earliest arrival?

Time versus structure (basic observations)

Centrality?



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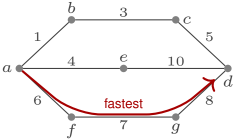
Optimal paths?

[Bui-Xuan, Ferreira, Jarry, 2003]



Which path is optimal from a to d?

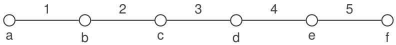
-min hop?

-earliest arrival?

-fastest traversal?

Time versus structure (basic observations)

Centrality?



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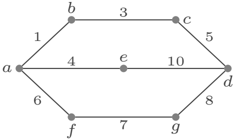


Which path is optimal from a to d?

-min hop?

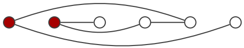
-earliest arrival?

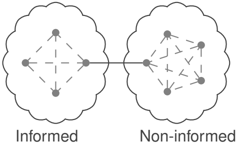
-fastest traversal?

Diameter of the snapshots versus propagation time

– Small diameter / Large propagation time ✓ $(\rightarrow)$

– Large diameter / Small propagation time ✓ $(\downarrow)$

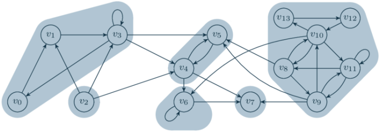




Connected components

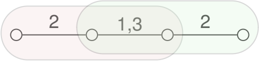
(Impact of non-transitivity)

In static graphs (directed or not)



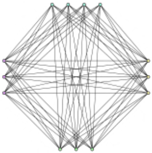
- Components define a partition
- Easy to compute

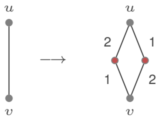
In temporal graphs



- Maximal components may overlap
- Can be exponentially many

TEMPORAL COMPONENT is NP-hard! (from CLIQUE) [Bhadra, Ferreira, 2003]





- Replace edges with semaphore gadgets
- Cliques $\Longleftrightarrow$ temporal components

Spanning trees

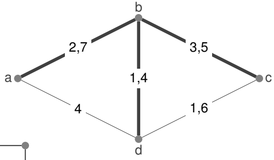
In static graphs

- Existence is guaranteed
- Size is always $n-1$

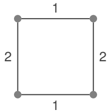
Temporal spanning tree ?

Input: A temporal graph $\mathcal{G} \in$ TC.

Goal: Find a spanning tree S of the *footprint*, so that $\mathcal{G}[S] \in$ TC.



Does not always exist:



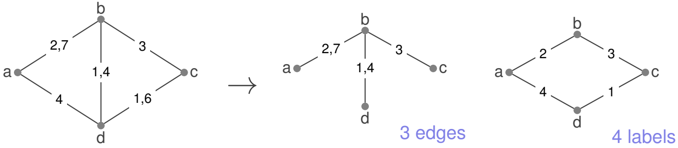
In fact, NP-hard to decide! [Casteigts, Corsini, 2024]

In search of the lost tree

Trees are not guaranteed, what could replace them?
→ Small reachability-preserving subgraphs.

Temporal spanners

Input: a temporal graph $\mathcal{G} \in$ TC
Output: $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathcal{G}' \in$ TC
Cost measure: # edges or # labels



Complexity:

- MIN-EDGE (and MIN-LABEL): APX-hard for simple, non-proper, non-strict [Axiotis, Fotakis, 2016]
- MIN-LABEL: APX-hard for non-simple, non-proper, strict [Akrida, Gasieniec, Mertzios, Spirakis, 2017]
- MIN-EDGE: NP-hard for non-simple, proper [Casteigts, Corsini, 2024]

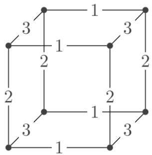
How about structural results?
→ Let's focus on **happy** temporal graphs (simple and proper)

Approved by
Pharrell W.:

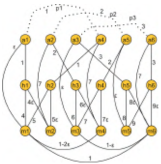
Structural results

Given $\mathcal{G} \in$ TC, what guarantee on the size of a spanner $\mathcal{G}' \subseteq \mathcal{G}$?

Note: The absolute minimum is $2n - 4$ [Bumby, 1979 (gossip theory)]



- Are spanners of size $O(n)$ always guaranteed?
  $\rightarrow$ Nope, hypercubes may fail [Kleinberg, Kempe, Kumar, 2000]
- Are spanners of size $o(n^2)$ always guaranteed?
  $\rightarrow$ Not even! [Axiotis, Fotakis, 2016]



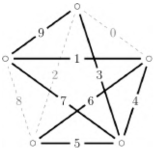
Any positive results?

Good news 1 (probabilistic): [C., Raskin, Renken, Zamaraev, FOCS 2021]:

- Nearly optimal spanners (of size $2n + o(n)$) almost surely exist in **random** temporal graphs, **as soon as** the graph becomes TC!

Good news 2 (deterministic): [C., Peters, Schoeters, ICALP 2019]:

- Spanners of size $O(n \log n)$ always exist in temporal **cliques**

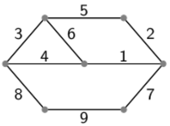


Connectivity in random temporal graphs

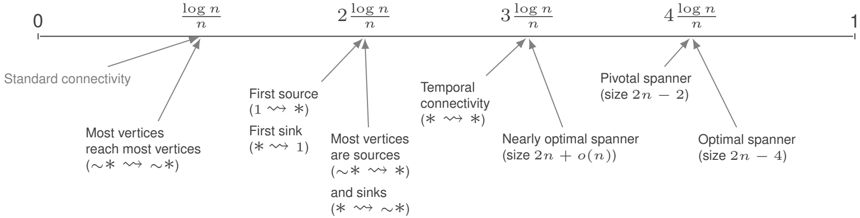
(C., Raskin, Renken, Zamaraev, 2021)

Temporal Erdös-Rényi graphs:

1. Pick an Erdös-Rényi $G \sim G_{n,p}$
2. Permute the edges randomly, interpret as (unique) presence time



Timeline for p (as $n \to \infty$):

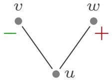


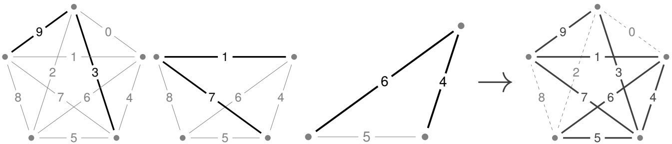
All the thresholds are sharp.

(sharp: $\exists \epsilon(n) = o(1)$, not true at $(1 - \epsilon(n))p$, true at $(1 + \epsilon(n))p$)

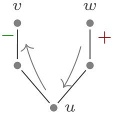
$O(n \log n)$ spanners in temporal cliques

Dismountability: Recursive construction of a spanner based on this pattern $\rightarrow$

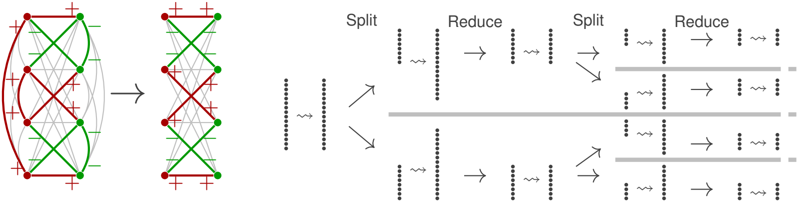




If the pattern is not present, try a 2-hop version $\rightarrow$



If none of the two patterns are present, we have a lot of structure.
(verbal explanations)

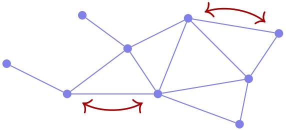


$$cost(n) = 2 \cdot cost(n/2) + O(n) = \boxed{O(n \log n)}$$ (Master's theorem)

Refs: [Casteigts, Peters, Schoeters, ICALP 2019] / [Angrick et al., ESA 2024] / [Carnevale, Casteigts, Corsini, SAND 2025]

Distributed Algorithms

(Think globally, act locally)



Collaboration of distinct entities to perform a common task.

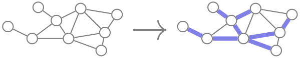
No centralization available, interactions among neighbors.

Theoretical aspects of collective intelligence.

Examples of problems:

Broadcast

Spanning tree

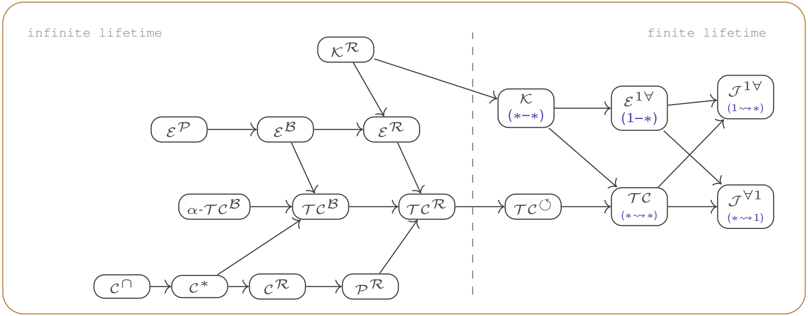


Election

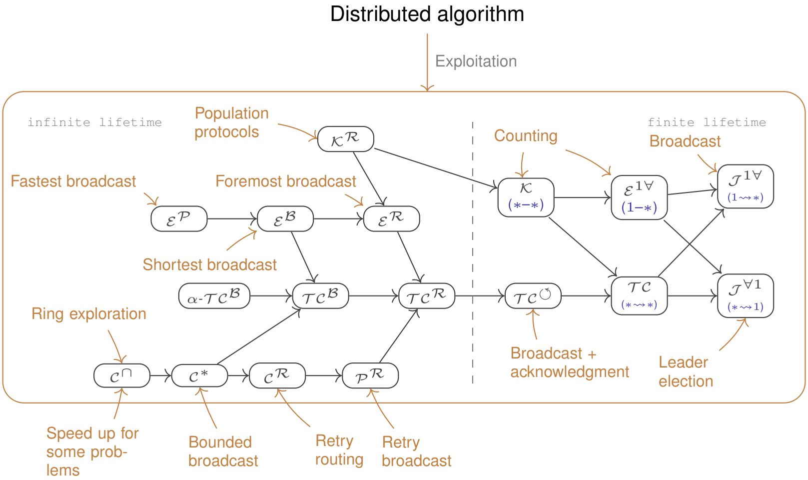
Counting

Consensus, naming, routing, exploration, coloring, dominating sets, ...

Some classes of temporal graphs



Some classes of temporal graphs



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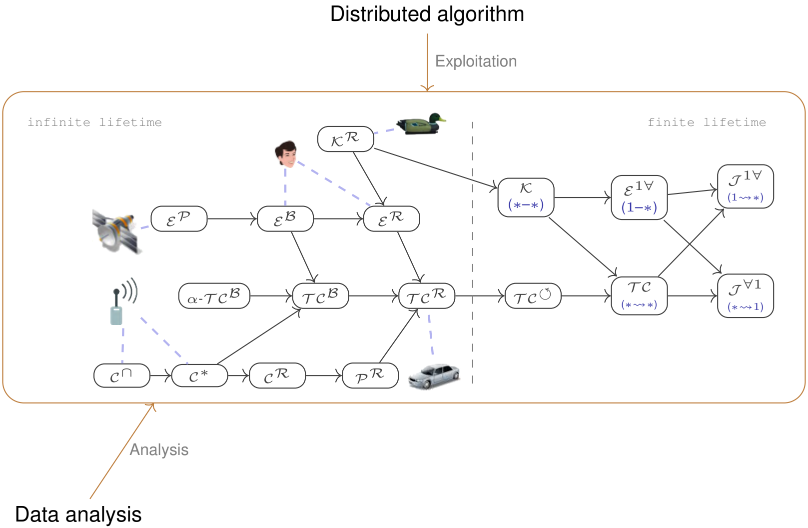

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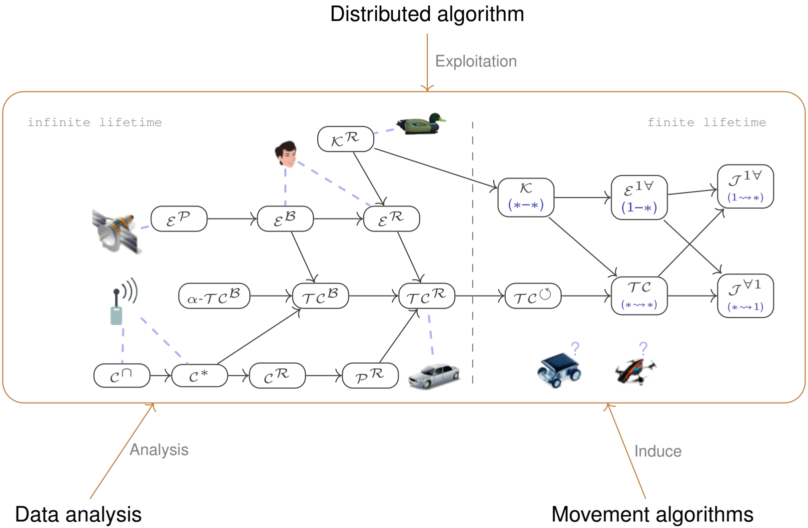


Open questions and research challenges

General challenges

- Few structural results known in temporal graph theory.
- Few positive results as well on the algorithmic side.
- Inappropriate measures of complexity (e.g. FPT in what parameters?)
- Are there non-trivial applications within TCS itself?

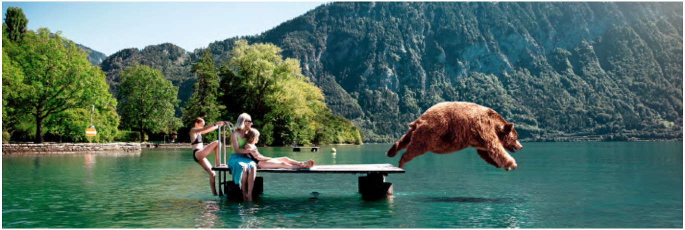
Specific to spanners

- Complexity of MIN-SPANNER in happy graphs?
- Do cliques admit spanners of size $O(n)$?
- How about other families?

Progresses guided by applications

- The studied problems are already often motivated by application
- How about the properties of the input graphs?
- What families of temporal graphs correspond to real world problems?

Thanks!



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