

The Saga of the Four-Color Theorem

(+ other facts about graph coloring)

Arnaud Casteigts

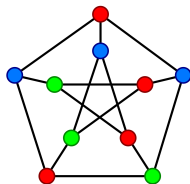
University of Geneva

ALGO seminar

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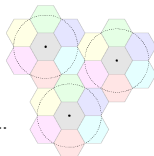
Graph coloring

Goal: Assign a color to every vertex such that adjacent vertices have different colors.



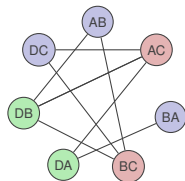
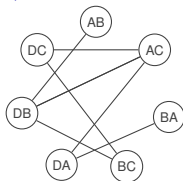
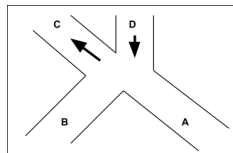
Many applications (e.g. telecom)

- ▶ Wireless communications
- ▶ In general, mutual exclusion, scheduling, ...
- ▶ Occupy a 5-y.o. kid



Graph coloring (2)

Example: traffic lights (credit: Eric Sopena)



1) Create a conflict graph G of the trajectories.

2) Color G

3) We obtain 3 *color classes*: 1 : {AB, DC, BA}, 2 : {DB, DA}, 3 : {AC, BC}.

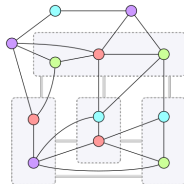
→ All trajectories in the same class can have green light at the same time.

Further examples: time tables; altitude of aircrafts; anything to be optimized against *conflicts*.

Chromatic number $\chi(G)$

$\chi(G)$ = minimum number of colors needed in G .

- ▶ At least the size of any *clique* in G
- ▶ Hadwiger's conjecture (1943):
At most the size of a clique *minor* in G .



On the algorithmic side

Complexity of k -coloring

- ▶ 2-COL is linear (if and only if bipartite graph)
- ▶ 3-COL is NP-hard (reduction from SAT)
- ▶ 4-COL is NP-hard (reduction from 3-COL)
- ▶ k -COL is NP-hard (reduction from $(k-1)$ -COL)



[Garey, Johnson, Stockmeyer, 1976]

(drawing: Yu Cheng)

The First-Fit algorithm

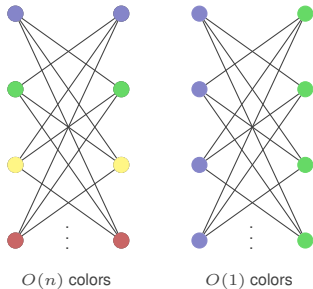
For each vertex in G :

Try color 1, then 2, then 3...

Very fast, but arbitrarily far from optimum
(if we pick the vertices in bad order)

Still, essentially the best we can do!

(No $n^{1-\epsilon}$ approx in polynomial time... [Zuckermann, 2007])

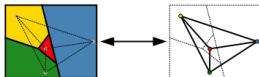


The four color theorem

Every *planar* graph is four colorable (*planar* = can be drawn without crossing edges).

Timeline:

- 1852 Francis Guthrie (botanist) notices that **four** colors are enough to color the map of England's counties.
- 1879 Kempe proves the conjecture.
- 1880 Tait formulates it in terms of **planar graphs**, and gives a different proof.
- 1890 Heawood finds a bug in Kempe's proof and adapts it to prove that **five** colors are enough.
- 1891 Petersen finds a bug in Tait's proof.
- 1960s Heesch starts using computers to search for a proof
- 1976 Appel and Haken **succeed!**
→ reduction from ∞ to 1834 possible configurations, all **checked by computer**.
- 1996 Robertson, Sanders, Seymour reduce it to 633 configurations.
- 2005 Gonthier certifies the proof using **Coq**. □

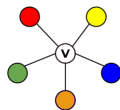


Proof of the 5 color theorem

Theorem: planar graphs are 5 colorable.

Warm-up: what about 6 colors first?

- ▶ Euler's formula: in planar graphs, $v - e + f = 2$ ($v = \#$ vertices, $e = \#$ edges, $f = \#$ faces)
- ▶ Implies (not immediate) that every planar graph has a vertex of degree ≤ 5
- ▶ Recursive algorithm:
 1. Find a vertex v of degree ≤ 5
 2. Color $G \setminus v$
 3. Give an available color to v (guaranteed by its degree)Base case: If G has ≤ 5 vertices, give a different color to each vertex.



5 colors: Kempe's chains

Similar ideas as 6 colors, with an additional trick (by Kempe, 1879).

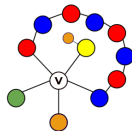
Step 2 becomes:

Color $G \setminus v$, then tweak the coloring so that the neighbors of v use at most 4 colors.

This is indeed always possible!

There must exist two colors which do not induce a connected component

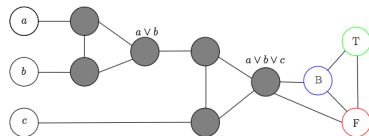
→ flip one of the components, one color is freed.



4 colors?

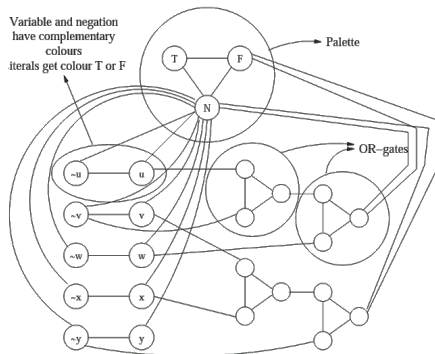
Not for today :-)

Reduction from 3-SAT to 3-COL



(credit: Lalla Mouatadid)

$$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$$



(credit: Igor Potapov)